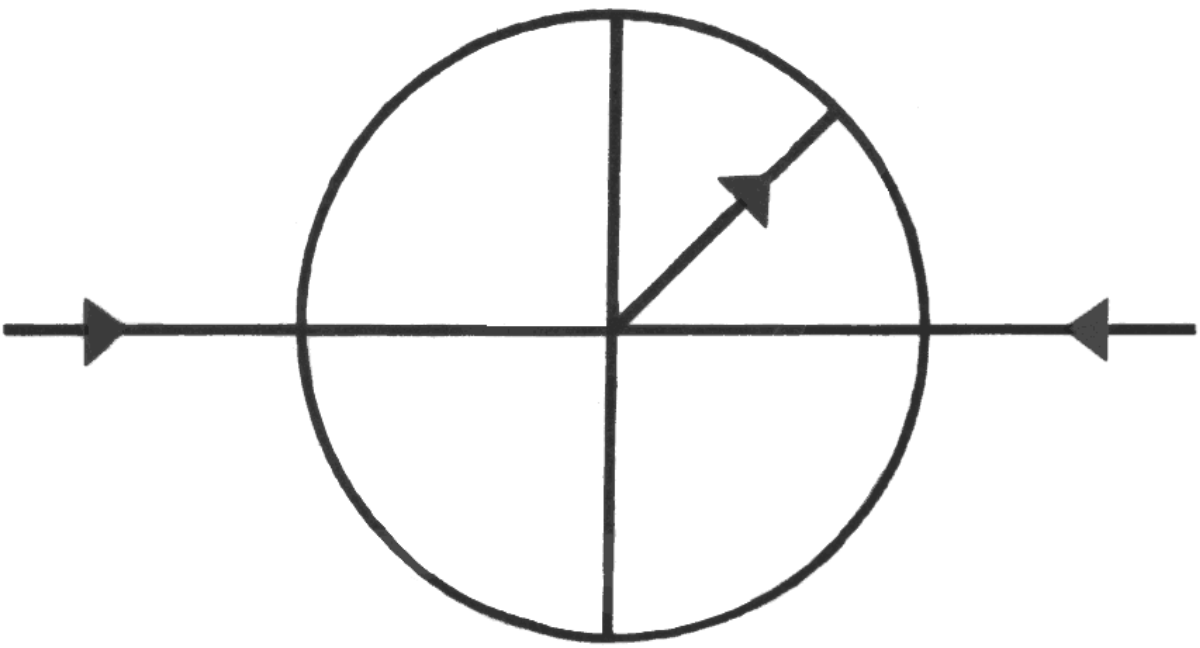
METHODS OF STRUCTURAL ANALYSIS



Negussie Tebedge

Methods of Structural Analysis provides the student of engineering with a concise working description of the classical methods of structural analysis and introduces the concept of matrix formulations of structures.

The basic principles of structural analysis are brought out by a simplified, coherent approach aided by the use of numerous diagrams and worked examples.

Students undertaking courses in the theory of structures and structural analysis will find this book extremely useful either as a main text, or as a supplement to other works in the field.

For a note on the author, please see the back flap.

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METHODS
OF
STRUCTURAL ANALYSIS

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Methods of Structural Analysis

NEGUSSIE TEBEDGE

Associate Professor of Civil Engineering
Addis Ababa University

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To my parents

Contents

Preface

This textbook has been compiled from a set of lecture notes developed while teaching courses in the theory of structures to civil engineering students at Addis Ababa University during the past seven years. The book is primarily intended for use as a text for instruction and contains sufficient material for a two-semester course in theory of structures. It may also be useful to the structural engineer who wishes to strengthen his background in structural mechanics.

The purpose of this book is to present a balanced treatment of the fundamental principles of structural mechanics, with their applications to the analysis of structural systems and their components. The coverage is selective, to allow a thorough treatment of the most common and useful analytical methods of structural analysis.

An attempt is made to present the subject matter in a unified, coherent and easy-to-understand manner which brings out the basic principles underlying the field of structural theory. The book is illustrated with ample example problems, to which solutions are presented to demonstrate the various methods, and also to widen the scope of the subject covered by the text.

The author is indebted to the authors of the many books he has freely consulted in the preparation of this work. The author also wishes to acknowledge his debt to all his students who, over the years, checked out the examples and assignment problems.

NEGUSSIE TEBEDGE
Addis Ababa
June, 1982

1. Introduction

1.1 STRUCTURAL ANALYSIS

Structural analysis is the process of determining the response of a structure due to specified loadings in order to satisfy essential requirements of function, safety, economy and sometimes aesthetics. This response is usually measured by calculating the reactions, internal forces of members, and displacements of the structures.

Structures may be classified into two general categories: ***statically determinate*** and ***statically indeterminate***. A structure which can be completely analysed by means of statics alone is called statically determinate. It then follows that a statically indeterminate structure is one which cannot be analysed by means of statics alone.

There are specific advantages and disadvantages in using one type of structure over the other. The primary advantage of a statically indeterminate structure is that it will generally have lower bending moment and shear force than a comparable determinate structure. Another advantage of a statically indeterminate structure is that it is generally stiffer for a given weight of material than a comparable determinate structure. Both of these advantages are a result of continuity of structural members acting to reduce stress intensities and displacements. A statically indeterminate structure can often furnish a compensation by redistribution within the structure in the case of overloads. On the other hand, however, indeterminate structures introduce computational difficulty in establishing the required equations. Another disadvantage is that indeterminate structures are, in normal cases, internally stressed due to differential settlement of supports, temperature changes and errors in the fabrication of members.

1.2 STATICAL INDETERMINACY

Consider a structure in space subjected to non-coplanar system forces. For the structure to be in equilibrium, the components of the resultants in the three orthogonal directions must vanish. This condition constitutes the six equations of equilibrium in space which are written as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$
$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \qquad [1.1]$$

For a structure subjected to a coplanar force system, only three of the six equations of equilibrium are applicable. The three equations of equilibrium in the xy plane are

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0 \qquad [1.2]$$
$$\Sigma M_z = 0$$

When a structure is in equilibrium, each member, joint, or segment of the structure must also be in equilibrium and the equations of equilibrium must also be satisfied. As discussed earlier, a structure which can be analysed by means of the equations of equilibrium alone is statically determinate. This book deals with statically indeterminate structures in which the structures cannot be analysed by the equations of equilibrium alone.

When a structure is statically indeterminate, there is some freedom of choice in selecting the member or reaction to be regarded as redundant. When the reaction is taken as the redundant, the structure is said to be *externally indeterminate*. On the other hand, when the member itself is regarded as the redundant, the structure is said to be *internally indeterminate*. It is also possible that the structure may have a combination of external and internal indeterminacy.

The question of identifying external or internal indeterminacy is largely of academic interest. What is of primary importance in the analysis of indeterminate structures is to know the degree of total indeterminacy. Nevertheless, a separate discussion of external and internal indeterminacy is desirable as a method to evaluate the degree of total indeterminacy.

(a) External Indeterminacy If the total number of reactions in a structure exceeds the number of the equations of equilibrium applicable to the structure, the structure is said to be externally indeterminate. The structures shown in Fig. 1.1 are examples of external indeterminacy. Each of the structures has five reaction components. Since there are only three equations of equilibrium, there are two extra reaction components that cannot be determined by statics. The number of unknown reactions in excess of the applicable equations of equilibrium defines the degree of indeterminacy. Thus the structures of Fig. 1.1 are indeterminate to the second degree. An alternative approach to determine the degree of indeterminacy would be to remove selected redundant reactions until the structure is reduced to a statically determinate and stable base or primary structure.

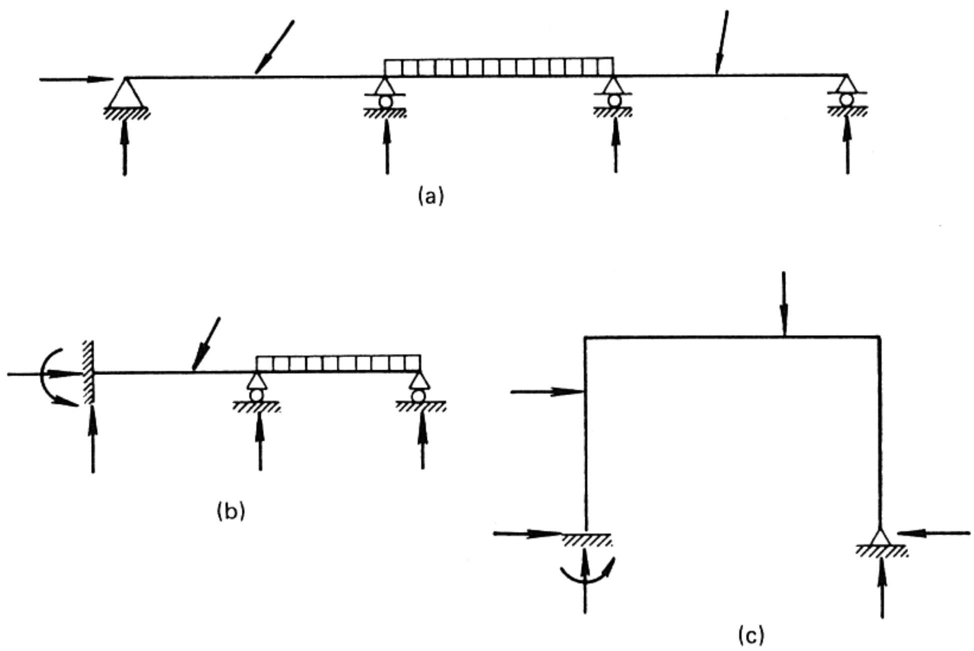


Figure 1.1

(b) Internal Indeterminacy A structure is internally indeterminate when it is not possible to determine all internal forces by using the three equations of static equilibrium. For the great majority of structures, the equation of whether or not they are indeterminate can be decided by inspection. For certain structures this is not so, and for these types rules have to be established. The internal indeterminacy of trusses will be first considered, and then that of continuous frames.

It is evident that any truss developed by using three bars connected at three joints in the form of a hinged triangle, and then using two bars to connect each additional joint, forms a stable and determinate truss. This is because the shape of the triangle cannot be changed without changing the length of any of the members. For stable and determinate trusses, built up as an assemblage of triangles, there are two conditions of equilibrium for each joint, so that if there are j joints, m members and r reaction components, a test for statical determinacy is:

$$2j = m + r \qquad [1.3]$$

In this equation, the left-hand side represents the total possible number of equations of equilibrium, while the right-hand side represents the total number of unknown forces.

The above equation is usually written in the form

$$m = 2j - r \qquad [1.4]$$

If there are more members than are indicated by the equation, then the structure is statically indeterminate; whereas if it has fewer members it is unstable. Caution must be exercised in applying the above equation because of the fact that the fulfilment of this equation is a *necessary* condition but not *sufficient* for internal stability of trusses. This may be summarised as

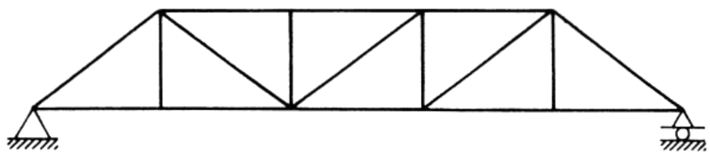
$m = 2j - r$ (determinate if stable)

$m > 2j - r$ (indeterminate if stable)

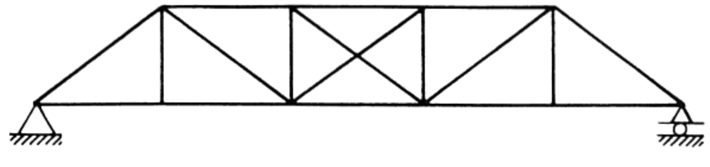
$m < 2j - r$ (unstable)

The truss in Fig. 1.2(a) has $m = 17, j = 10$ and $r = 3$. Application of [1.4] gives $(10 \times 2) - 3 = 17$ members, thus the structure is statically determinate. Referring to Fig. 1.2(b), there are 18 members, or one more member than is needed for a determinate structure; thus, the additional diagonal member is redundant and the truss is indeterminate to the first degree. Figure 1.2(c) represents the omission of one diagonal member, keeping the same total number of bars, $m = 17$. Again the condition equation is satisfied. However, inspection of the truss indicates that the structure is unstable with one panel free to collapse, thus causing the entire truss to collapse. Hence, satisfaction of the above equation is not a sufficient condition for internal stability of trusses. Inspection of the structure and consideration of stress paths are more reliable approaches to settle the question of stability and internal indeterminateness of trusses.

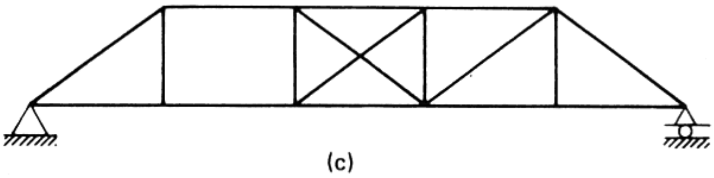
An alternative approach to determine the degree of indeterminacy of trusses



(a)



(b)



(c)

Figure 1.2

is by removing the redundant quantities until a determinate and stable base structure remains.

The number of rigidly jointed frames are subject to shearing forces, bending moment and axial force, so that there are three unknown internal forces for each member, or a total of $3m$ unknown components. Moreover, at each joint three equations of equilibrium can be written, giving $3j$ equations in all. Therefore for a statical determinacy, it is necessary that

$$3j = 3m + r \qquad [1.5]$$

or that the number of redundants n is given by

$$n = 3m + r - 3j$$

When there is a roller or pin support, the degree of indeterminacy is reduced by one or two, respectively, for each support.

An alternative approach, which in this case may be considered more instructive, is the method by inspection where the structure is cut until it becomes a determinate and stable base structure. Consequently, the total number of released internal force components corresponds to the degree of indeterminacy.

1.3 KINEMATIC INDETERMINACY

When a structure is subjected to a system of forces, the overall behaviour of the members of the structure may be defined by the displacement of the joints. The joints undergo displacements in the form of translation and rotation. A system of joint displacements is known to be independent if each displacement can be varied arbitrarily and independently of the other displacements. The number of independent joint displacements that serve to describe all possible displacements of a structure is known as the number of ***degrees of freedom*** or ***degree of kinematic indeterminacy***.

In determining the degree of kinematic indeterminacy, attention is focused on the number of independent displacement degrees of freedom that the structure possesses. If a structure has n degrees of freedom, that is, n number of independent displacement quantities required to describe all possible displacements for any loading condition, the structure is said to be kinematically indeterminate to the nth degree. When these displacements are set to zero, the structure then becomes *kinematically determinate*.

Consider, for example, the rigid-jointed plane frame shown in Fig. 1.3, which is fixed at supports A and C and has a hinged support at D. Assuming that the axial deformations are negligible, there will be no axial displacements in the frame and the only unknown displacements are the joint rotations θ_B and θ_D at joints B and D, respectively. Since these displacements are independent of one another, the degree of kinematic indeterminacy of this structure is two.

It is observed that the degree of statical indeterminacy of the frame of Fig. 1.3 is four since there are a total of seven possible unknown reactions and three equations of equilibrium. If, for instance, the fixed support at C is replaced by a hinge, the degree of statical indeterminacy is reduced to three since an additional equilibrium condition is introduced. However, the kinematic indeterminacy is increased by one since an independent rotation at C now becomes possible. In general, an introduction of a displacement release decreases the statical indeterminacy and increases the kinematic indeterminacy.

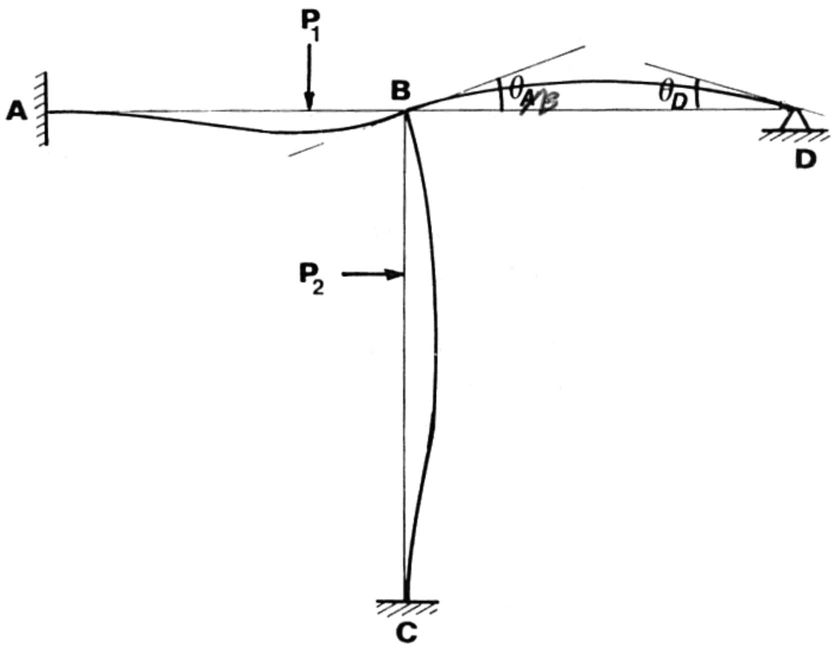


Figure 1.3

1.4 METHODS OF STRUCTURAL ANALYSIS

The objective of structural analysis is to study the response of a structure to specified loadings after determining the external reactions and internal stress resultants. The forces determined must satisfy the conditions of equilibrium and the displacements produced by these forces must be compatible with the continuity of the structure and the support conditions. In determining the unknown forces in a statically indeterminate structure, the equations of equilibrium are not sufficient, and additional equations must be formulated based on compatibility of displacements. These supplementary equations that ensure the compatibility of the displacements with the geometry of the structure are known as the *compatibility* conditions.

Two general methods of analysis are available for the solution of statically indeterminate structures. The first is the *force* or *flexibility* method. This method is simple and conceptually straightforward to understand and provides an effective method for certain types of structures. In this method the structure is made statically determinate by providing a sufficient number of releases by removing the redundant forces. Due to the given loading condition the primary structure undergoes inconsistency in geometry which must then be corrected by applying the redundant forces such that compatibility conditions throughout the structure are established. This method is sometimes referred to as the *compatibility* method.

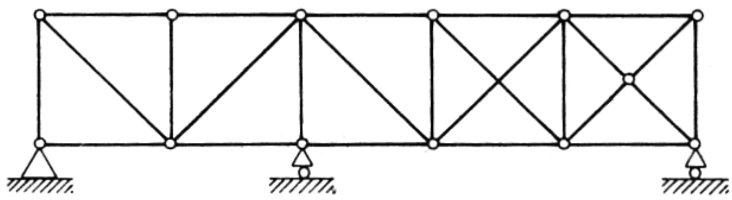
The second method of analysis of statically indeterminate structures is the *displacement* or *stiffness* method. This method is also simple and straightforward and provides an effective method for certain classes of structure. In this method, restraints are imposed to prevent displacement of joints until the structure becomes kinematically determinate and the forces required to produce the restraints are evaluated. Displacements are then permitted to take place at the restrained joints until the imposed restraining forces have been removed such that equilibrium conditions throughout the structure are established. This method is also known as the *equilibrium* method.

Either the force or the displacement method can be used to analyse any structure. The choice of the method of analysis, either force or displacement, depends largely on the degree of statical or kinematic indeterminacy. In both methods, the analysis generally involves the solution of a system of simultaneous equations where the number of unknown variables must be equal to the degree of indeterminacy. If manual calculations are to be adopted, it would be logical to use the method that produces the smaller set of simultaneous equations.

1.5 PROBLEMS

Determine the degree of statical indeterminacy of the structures shown below.

1.1



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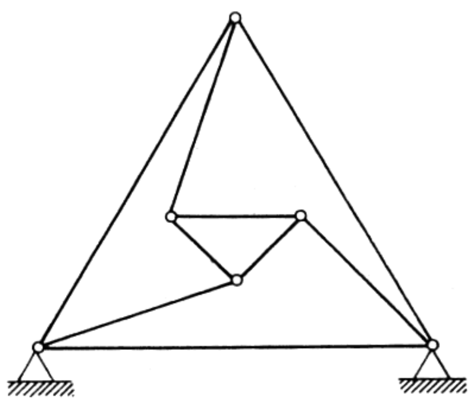

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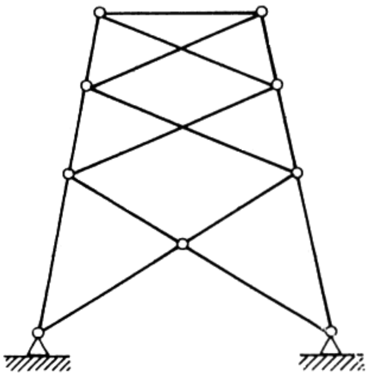

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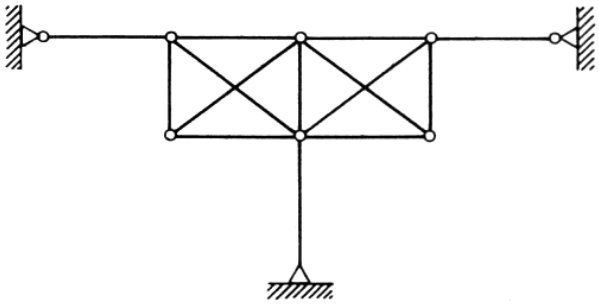

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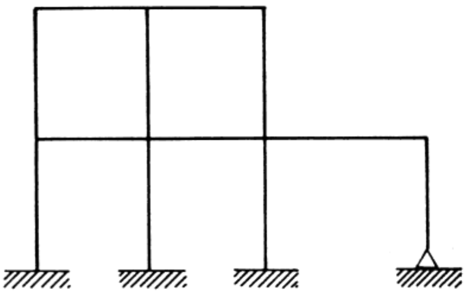

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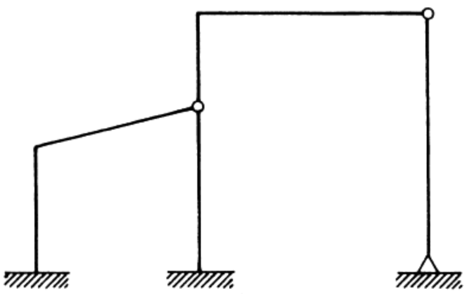

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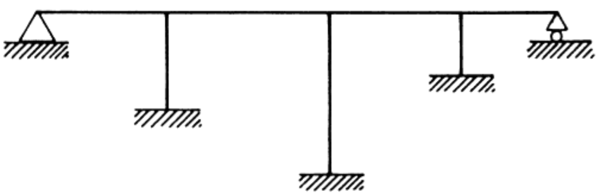
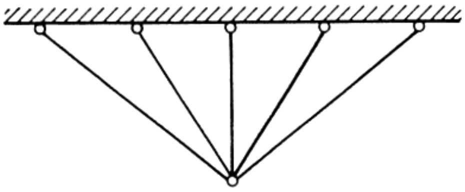


Determine the degree of statical and kinematical indeterminacy of the structures shown below.

1.8



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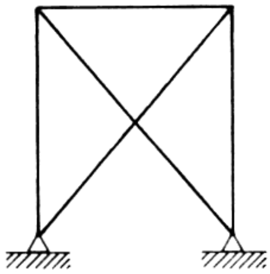

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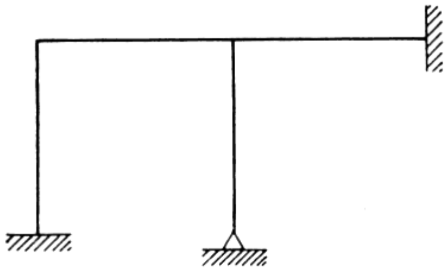

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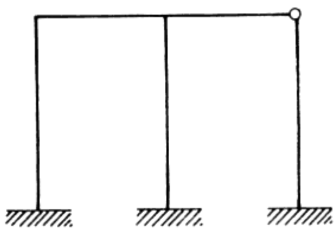

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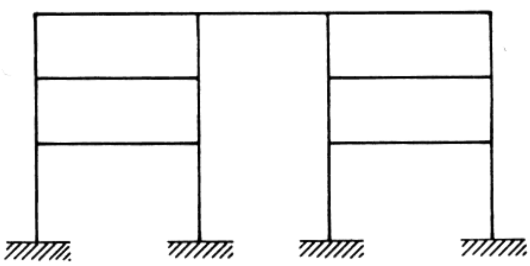

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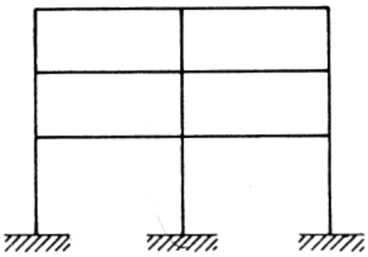

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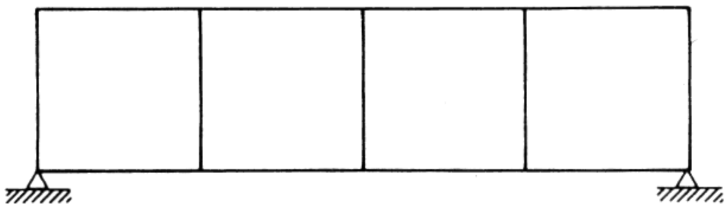


2. Methods of Consistent Displacements

2.1 INTRODUCTION

For statically indeterminate structures there will be an indefinite number of combinations of redundant forces which will satisfy equilibrium conditions. However, among them there will be only one set of values that will simultaneously satisfy the requirements of equilibrium and compatibility. Compatibility places constraints on the displacements of a structure to ensure continuity and that the structure conforms to the displacement boundary conditions prescribed by the supports.

The methods of consistent displacement are based on the concept of equilibrium of forces and compatibility of displacements which may be stated as follows: Given a set of forces applied on a statically indeterminate structure, the reactions must assume such values that satisfy not only the conditions of static equilibrium with the applied loads but also the conditions of compatibility. The general method of consistent deformation is applicable for analysing all types of indeterminate structures. It is also applicable whether the structure is subjected to external loading, temperature changes, movements of supports, fabrication errors, or any other cause. Of course, there are other methods that are definitely superior for certain specific structures or loading conditions, but methods of consistent deformation are the most versatile and general.

2.2 ANALYSIS OF BEAMS

The principle of *consistent displacement* can best be illustrated by considering singly indeterminate structures. As a simple and classic example of this method consider a propped cantilever beam as shown in Fig. 2.1. The beam has three unknown reactions V_A, M_A and V_B and is therefore statically indeterminate to

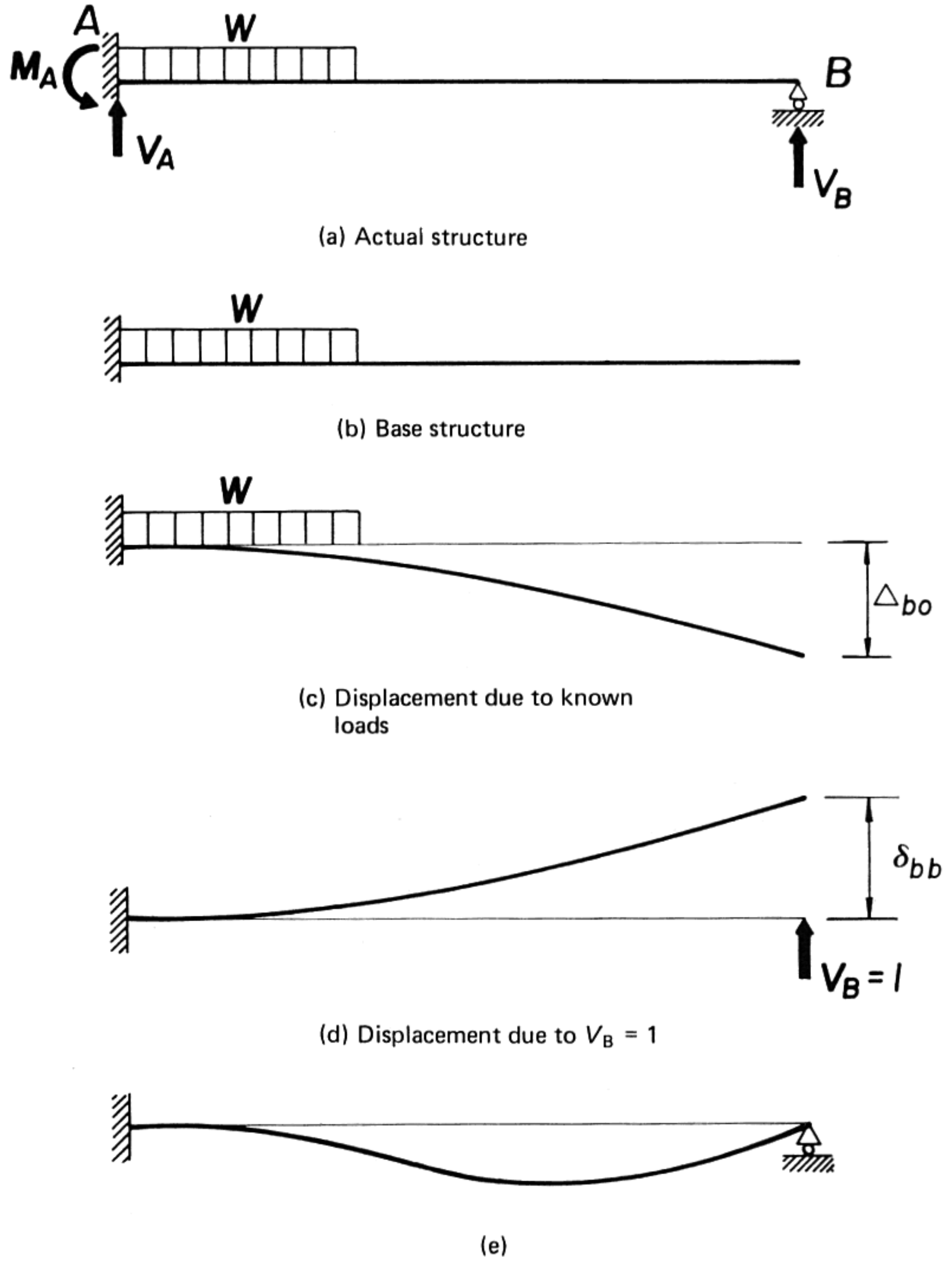


Figure 2.1

the first degree. Any one of the unknown reactions may be taken as the redundant. A stable and determinate primary structure may be formed by determinate primary structure by selecting as the redundant the vertical reaction at the right support, V_B, as shown in Fig. 2.1(b).

The displacement of the cantilever beam AB may be considered to consist of the superposition of two independent displacements:

Δ_{bo} = upward deflection at B of the base structure due to the known applied loads only

Δ_{bb} = upward deflection at B of the base structure due to the redundant V_B

It may be noted that it is not possible to evaluate Δ_{bb} prior to the evaluation of V_B. However, by applying the principle of superposition such that $\Delta_{bb} = \delta_{bb}V_B$, where

δ_{bb} = upward deflection at B of the base structure due to a unit upward load at B

then, the condition that the support at B is rigid requires its displacement, Δ_B, the algebraic sum of displacements due to the applied loads and the redundant, must be zero. This geometric condition, defined as the *equation of consistent deformation*, may be writtten as

$$\Delta_B = \delta_{bo} + V_B\delta_{bb} = 0 \qquad [2.1]$$

or

$$V_B = -\frac{\Delta_{bo}}{\delta_{bb}} = -\frac{\int \frac{Mmdx}{EI}}{\int \frac{m^2dx}{EI}} \qquad [2.2]$$

where M is the moment in the base structure due to the applied loads and m is the moment due to a unit load acting at B.

It is noted that if V_B acts in the same direction as Δ_{bb}, a negative value is obtained which indicates that the assumed direction is wrong. Conversely, a positive value for V_B indicates that the assumed direction is correct. In general, it must be noted that the magnitude of the true reaction V_B is that required to restore the end B of the beam to its original position level with A.

In a similar manner, if M_A which is the moment reaction at A is taken as the redundant, the applied loads will cause the tangent at A to rotate through an angle θ_{ao}. If the rotation due to a unit moment at A is taken as θ_{aa} the moment M_A necessary to rotate the tangent at A to the original horizontal position is

$$M_A = \frac{\theta_{ao}}{\theta_{aa}} = \frac{\int \frac{Mm_\theta dx}{EI}}{\int \frac{m_\theta^2 dx}{EI}} \qquad [2.3]$$

where m_θ is the moment due to a unit moment acting at A.

The analysis of beams of higher degree of indeterminacy follows closely the procedure described above. For a beam with n degrees of indeterminacy, n redundants are selected which will be removed from the structure and replaced by n effectively equivalent redundant forces $X_1, X_2, \ldots X_n$. All these redundant forces and the given external loads are applied on the base structure such that their magnitudes must cause the displacements at the points of application of the n redundants of the base structure to be equal to the displacement of the corresponding points on the actual structure.

Consider the four-span continuous beam of Fig. 2.2. The beam has three redundant reactions which can be chosen in a variety of ways, one of which is shown in Fig. 2.2(a).

At this stage it is convenient to follow a definite notation for the various redundant forces and displacements. The redundant forces X_a, X_b and X_c are recognised and identified by single subscripts which denote their point of application. The displacements are identified by double subscripts: the first subscript denotes the point on the base structure at which the displacement occurs, and the second subscript is used to denote the force producing the displacement. If, for example, the points A, B, C, etc. are the points on the base structure where the redundants occur, then,

X_a = the redundant force at point A

Δ_{ao} = the displacement in the base structure at point A in the direction of X_a, caused by the actual applied loads acting on the structure

δ_{aa} = displacement in the direction of X_a in the base structure caused by $X_a = 1$ and no other load acting

δ_{ab} = displacement in the base structure at A in the direction of X_a caused by $X_b = 1$ acting alone

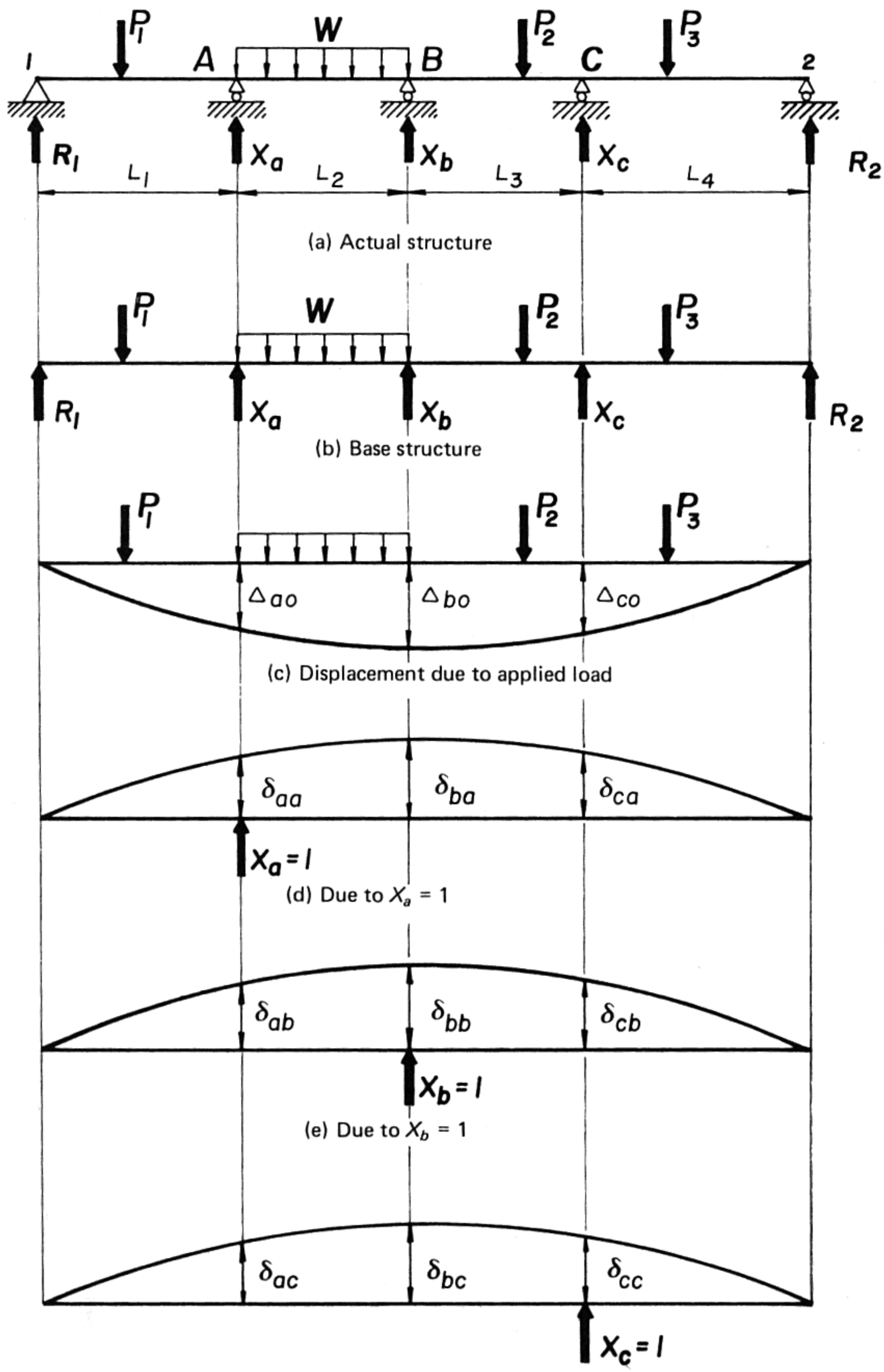
δ_{ac} = displacement in the base structure at A in the direction of X_a caused by $X_c = 1$ acting alone

Since the displacements at A, B, and C should be zero, the reactions X_a, X_b and X_c must have values such that compatibility condition is satisfied. Thus, using the above notation in the superposition equations, which gives as many equations as there are redundants, the equations may be written as follows:

$$\begin{aligned}\Delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} &= 0\\ \Delta_{bo} + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} &= 0\\ \Delta_{co} + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} &= 0\end{aligned} \qquad [2.4]$$

Since $\delta_{ab} = \delta_{ba}$, $\delta_{ac} = \delta_{ca}$, etc. by Maxwell's principle of reciprocal deflections, [2.4] may be written as

$$\begin{aligned}\Delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} &= 0\\ \Delta_{bo} + X_a\delta_{ab} + X_b\delta_{bb} + X_c\delta_{bc} &= 0\\ \Delta_{co} + X_a\delta_{ac} + X_b\delta_{bc} + X_c\delta_{cc} &= 0\end{aligned} \qquad [2.5]$$



(a) Actual structure

(b) Base structure

(c) Displacement due to applied load

(d) Due to $X_a = 1$

(e) Due to $X_b = 1$

(f) Due to $X_c = 1$

Figure 2.2

The equations can be written in the following matrix form:

$$\begin{bmatrix} \Delta_{ao} \\ \Delta_{bo} \\ \Delta_{co} \end{bmatrix} + \begin{bmatrix} \delta_{aa} & \delta_{ab} & \delta_{ac} \\ \delta_{ab} & \delta_{bb} & \delta_{bc} \\ \delta_{ac} & \delta_{bc} & \delta_{cc} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [2.6]$$

EXAMPLE 2.1 *Determine the reactions and support moment of the continuous beam shown in Fig. 2.3.*

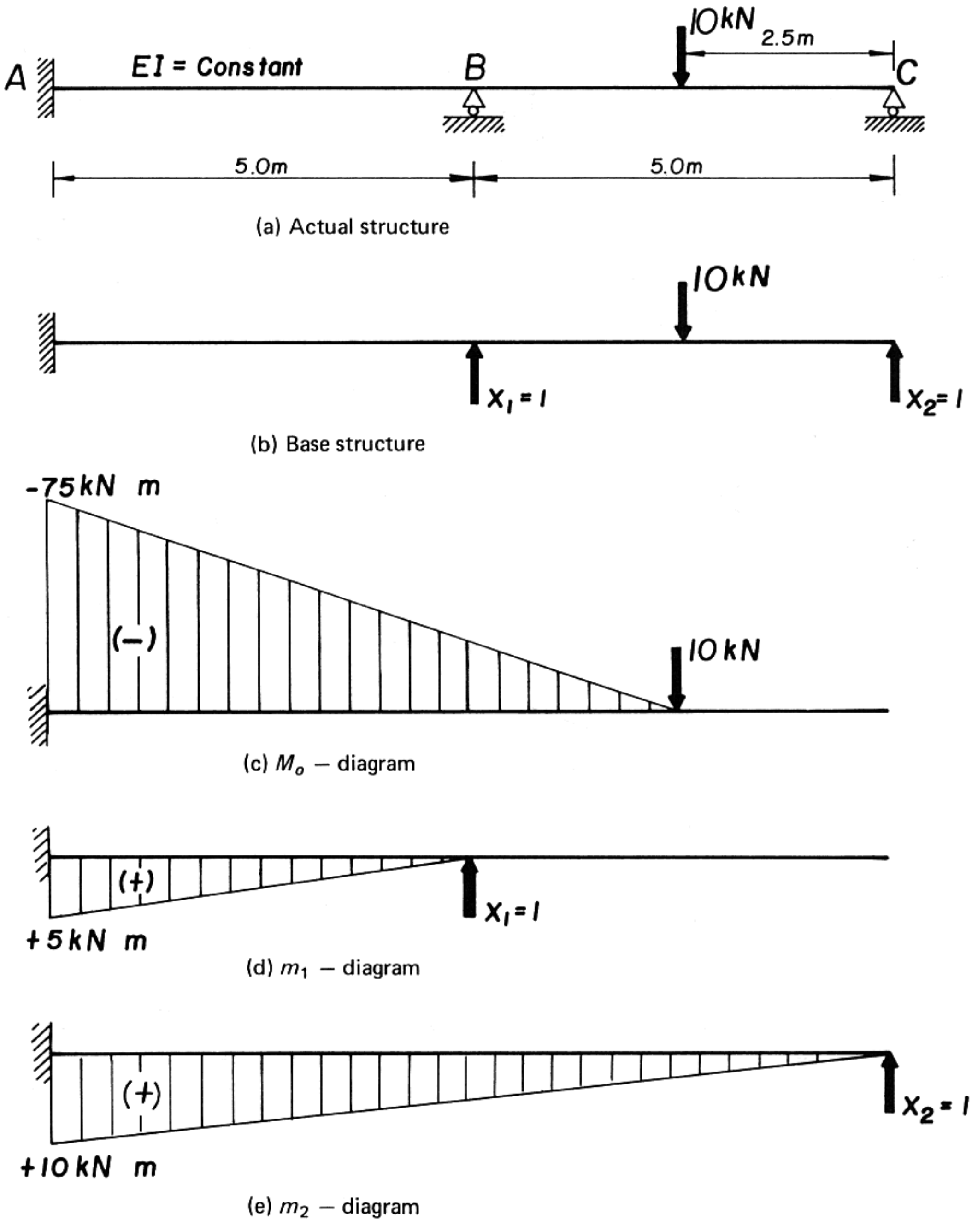


Figure 2.3

The beam is indeterminate to the second degree, and the redundants chosen are the reactions at B and C. The moment diagrams due to the applied load $X_1 = 1$ and $X_2 = 1$ are shown in Fig. 2.3(d) and (e) respectively.

The elastic equations are

$$\Delta_{10} + X_1\delta_{11} + X_2\delta_{12} = 0$$

$$\Delta_{20} + X_1\delta_{12} + X_2\delta_{22} = 0$$

The displacements are obtained by graphic multiplication method:

$$EI\Delta_{10} = \left(-\frac{5 \times 5}{2}\right)\left(\frac{17.5}{3} \times \frac{7.5}{7.5}\right) = -729.17$$

$$EI\delta_{11} = \left(\frac{5 \times 5}{2}\right)\left(\frac{2 \times 5}{3}\right) = 41.67$$

$$EI\delta_{12} = \left(\frac{5 \times 5}{2}\right)\left(\frac{25}{3}\right) = 104.17$$

$$EI\Delta_{20} = \left(-\frac{75 \times 7.5}{2}\right)\left(\frac{22.5}{3}\right) = -2109.38$$

$$EI\delta_{22} = \left(\frac{10 \times 10}{2}\right)\left(\frac{2 \times 10}{3}\right) = 333.33$$

Substituting the δ terms into the elastic equations:

$$-729.17 + 41.67X_1 + 104.17X_2 = 0$$

$$-2109.38 + 104.17X_1 + 333.33X_2 = 0$$

The solution of the simultaneous equations is

$$X_1 = 7.68 \text{ kN}$$

$$X_2 = 3.93 \text{ kN}$$

Thus,

$$R_B = 7.68 \text{ kN (upward)}$$

$$R_C = 3.93 \text{ kN (upward)}$$

From statics

$$R_A = 1.61 \text{ kN (downward)}$$

$$M_A = 2.70 \text{ kN (clockwise)}$$

Alternative Solution

The moment reaction and the vertical reaction at A are chosen as redundants. The moment diagram due to the applied loads $X_1 = 1$ and $X_2 = 1$ are shown in Fig. 2.4(d) and (e) respectively.

The displacements are

$$EI\Delta_{10} = \left(\frac{2.5 \times 2.5}{2}\right)\left(\frac{2}{3} \times 12.5\right) + \left(\frac{12.5 \times 2.5}{2}\right)\left(2.5 \times \frac{1}{3} \times 2.5\right)$$

$$= 78.13$$

$$EI\delta_{11} = 2\left(\frac{5.0 \times 5.0}{2}\right)\left(\frac{2}{3} \times 5\right)$$

$$= 83.33$$

$$EI\Delta_{12} = \left(-\frac{5.0 \times 5.0}{2}\right)(1.0) - \left(\frac{5.0 \times 5.0}{2}\right)\left(\frac{2 \times 1.0}{3}\right)$$

$$= -20.83$$

$$EI\Delta_{20} = -\left(\frac{12.5 \times 2.5}{2}\right)\left(\frac{2}{3} \times \frac{2.5}{5}\right) - \left(\frac{12.5 \times 2.5}{2}\right)\left(2.5 + \frac{2.5}{3}\right)\frac{1}{5}$$

$$= -15.63$$

$$EI\delta_{22} = (1.0 \times 5.0)(1.0) + \left(\frac{1.0 \times 5.0}{2}\right)\left(\frac{2}{3}\right)$$

$$= 6.67$$

Substituting into the elastic equations:

$$78.13 + 83.33X_1 - 20.83X_2 = 0$$

$$-15.63 - 20.83X_1 + 6.67X_2 = 0$$

The solution of the simultaneous equations is

$$X_1 = -1.61 \text{ kN}$$

$$X_2 = -2.70 \text{ kN m}$$

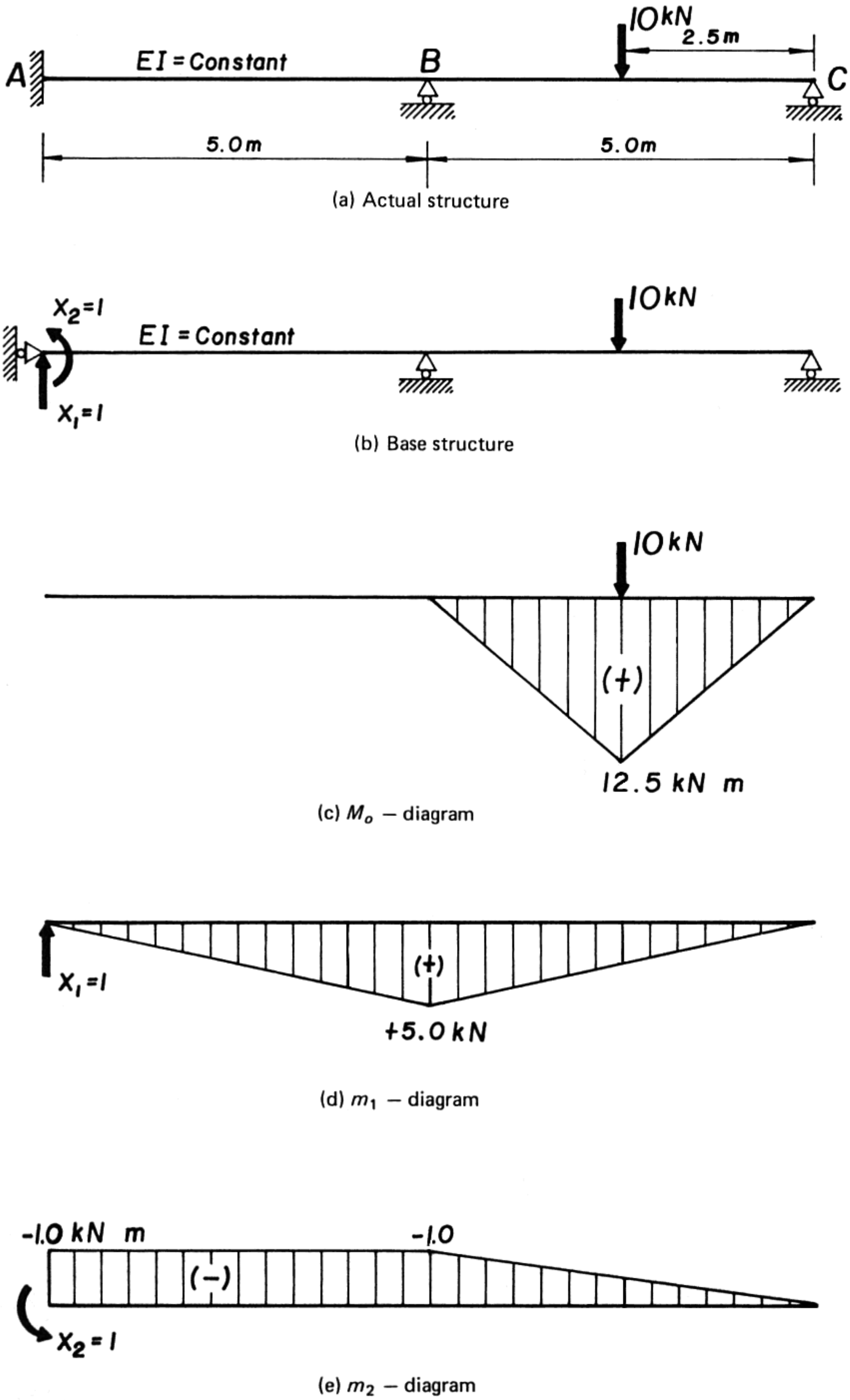
The reactions are

$R_A = 1.61$ kN (upward)

$M_A = 2.70$ kN m (clockwise)

$R_B = 7.68$ kN (upward)

$R_C = 3.93$ kN (upward)



10kN
2.5m
A
EI = Constant
B
C
5.0m
5.0m
(a) Actual structure
X2=1
10kN
EI = Constant
X1=1
(b) Base structure
10kN
(+)
12.5 kN m
(c) Mo – diagram
X1=1
(+)
+5.0kN
(d) m1 – diagram
-1.0 kN m
-1.0
(–)
X2 = 1
(e) m2 – diagram

Figure 2.4

2.3 ANALYSIS OF TRUSSES

A statically indeterminate truss with external redundant reaction or internal redundant member may be analysed by a procedure closely analogous to that followed in beams. The analysis of trusses with a redundant reaction consists of choosing a base structure by *removing* the redundant reactions. Acting on this base structure are the applied loading and the redundant reactions. Then the condition of compatibility is applied such that the displacements in the direction of the redundants become zero. In a similar manner, when the truss has redundant members, the base structure is obtained by *cutting* the redundant members and replacing it by a pair of forces and then applying the condition of compatibility. Take, for example, the truss shown in Fig. 2.5. The truss is internally indeterminate to the first degree.

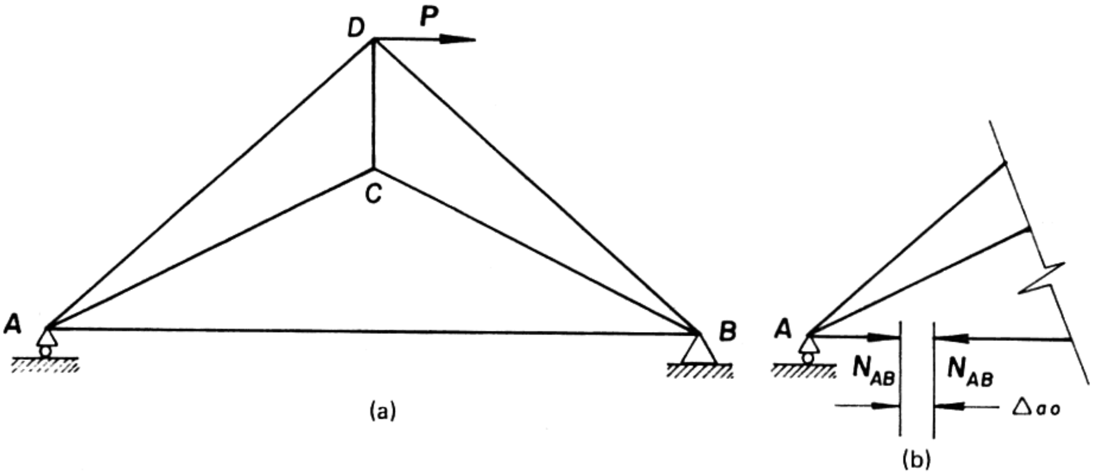


Figure 2.5

In this truss, any member may be considered redundant. Choosing member AB as the redundant, the redundant member is *removed* by cutting it at any section. Due to the effect of the external load P on the base structure, the two faces of the cut member AB will be displaced by Δ_{ao}. Now, applying a pair of forces N_{AB} as shown in Fig. 2.5(b), such that the relative displacement of the actual truss at the cut surface is zero, gives the following relationship:

$$\Delta_{ao} + N_{AB}\delta_{aa} = 0 \qquad [2.7]$$

where δ_{aa} is the relative displacement of the cut faces due to $N_{AB} = 1$. The internal force in the redundant member is

$$N_{AB} = -\frac{\Delta_{ao}}{\delta_{aa}} \qquad [2.8]$$

But from virtual work principle,

$$\Delta_{ao} = \Sigma \frac{NnL}{EA}$$
$$\delta_{aa} = \Sigma \frac{n^2L}{EA} \qquad [2.9]$$

where N = force in any member due to the external applied load acting on the base structure

n = force in any member due to a unit pair of forces applied at the cut faces of the member.

Thus

$$N_{\text{AB}} = \frac{\Sigma \dfrac{NnL}{EA}}{\Sigma \dfrac{n^2L}{EA}} \qquad [2.10]$$

Note that the summation in the denominator is taken over the whole truss, and the summation in the numerator applies only on the base structure.

The analysis of trusses of higher degree of indeterminacy follows closely the procedure described above. Consider, for example, the truss shown in Fig. 2.6 which is externally statically indeterminate to the second degree. If the supports at B and C are removed, a simple truss supported at A and D will be the basic determinate truss. The deflected bottom chord due to the applied loading is shown in Fig. 2.6(b). The displacements at B and C are determined from the expressions

$$\Delta_{bo} = \Sigma \frac{Nn_bL}{EA}$$
$$\Delta_{co} = \Sigma \frac{Nn_cL}{EA} \qquad [2.11]$$

Figure 2.6(b) shows the displacements at B and C due to a unit load applied at B, and in Fig. 2.6(c) due to a unit load applied at C. The vertical displacements are determined from the expressions

$$\delta_{bb} = \Sigma \frac{n_b^2L}{EA}$$
$$\delta_{bc} = \delta_{cb} = \Sigma \frac{n_bn_cL}{EA} \qquad [2.12]$$
$$\delta_{cc} = \Sigma \frac{n_c^2L}{EA}$$

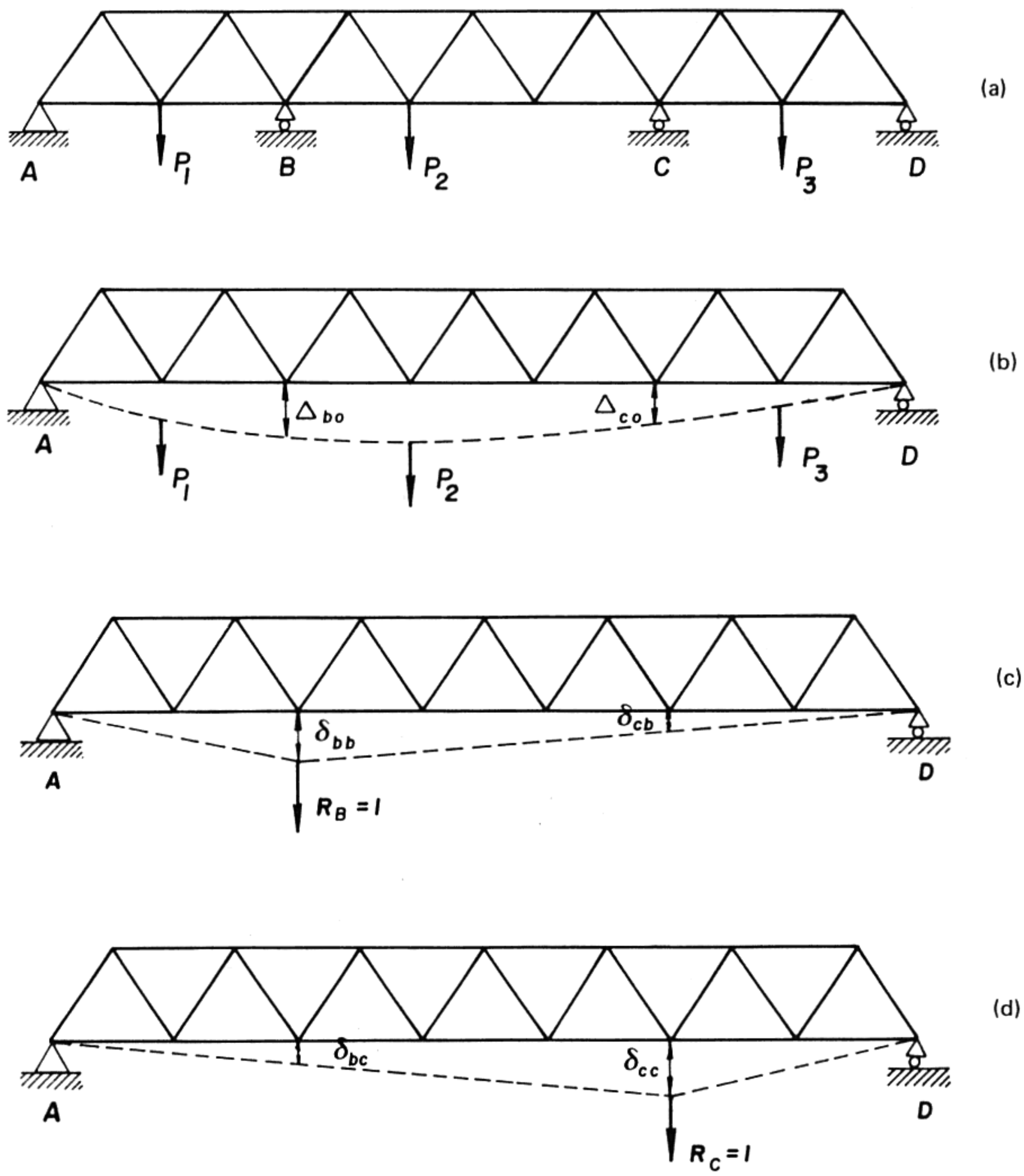


Figure 2.6

In the above expressions N stands for forces in the members due to the external applied loads on the base structure, and N_b and N_c are the forces in the members due to a unit load applied at B and C, respectively.

The conditions of compatibility required from which R_B and R_C can be determined are

$$\Delta_{bo} + R_b\delta_{bb} + R_c\delta_{bc} = 0$$
$$\Delta_{co} + R_b\delta_{bc} + R_c\delta_{cc} = 0 \qquad [2.13]$$

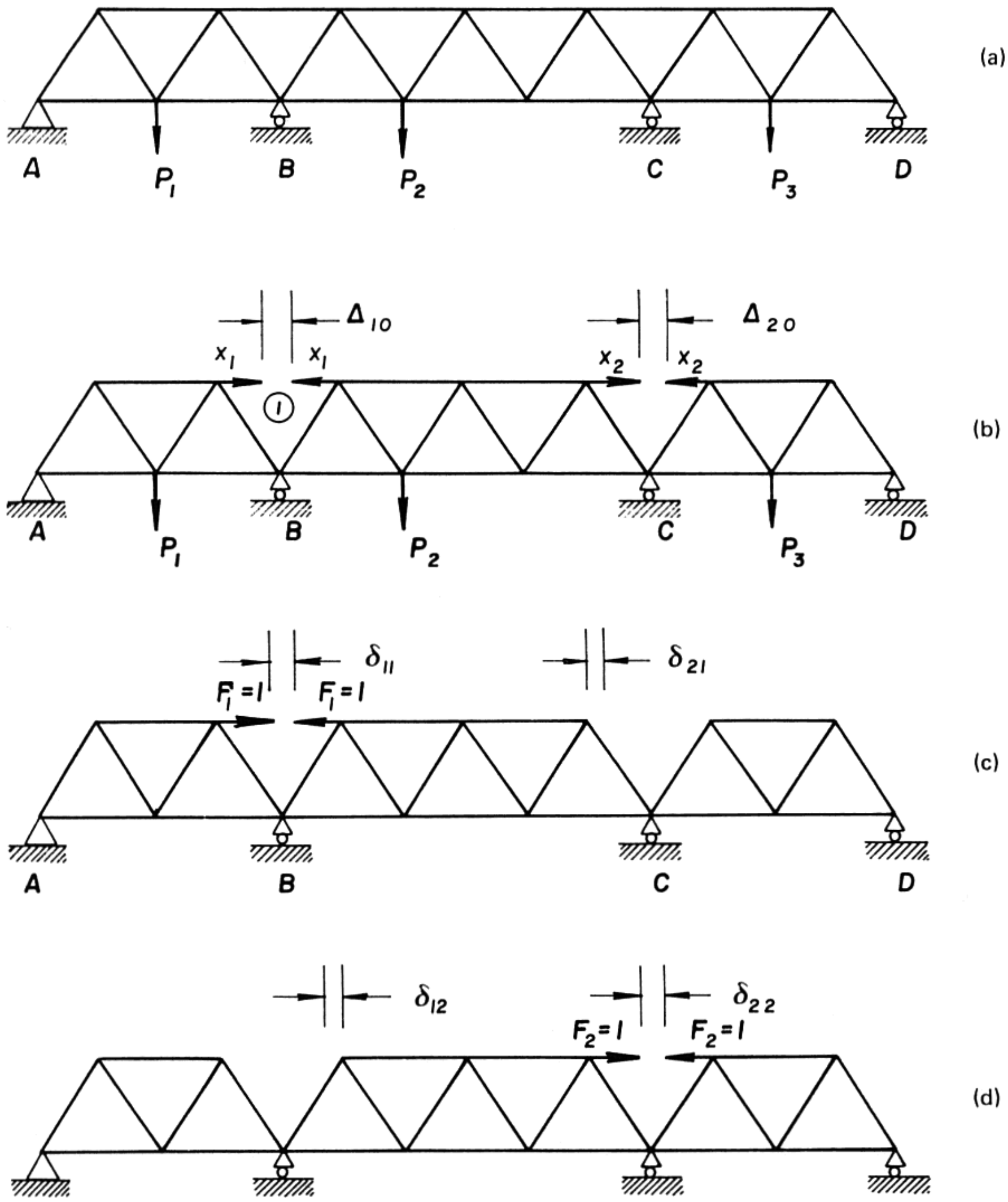


Figure 2.7

Consider again the truss shown in Fig. 2.6. If the redundants are taken as the bar forces X_1 and X_2 shown in Fig. 2.7(b), then the determinate truss is three independent simple span trusses. Due to the effect of the external loading on the base structure, the two faces of the cut members 1 and 2 will be displaced by Δ_{10} and Δ_{20}, respectively. After applying a pair of forces $F_1 = 1$ and $F_2 = 1$ as shown in Fig. 2.7(c) and (d), the corresponding relative displacements of the cut faces can be determined. The displacements are

$$\Delta_{10} = \Sigma \frac{Nn_1L}{EA}$$

$$\Delta_{20} = \Sigma \frac{Nn_2L}{EA}$$

$$\delta_{11} = \Sigma \frac{n_1^2L}{EA} \qquad [2.14]$$

$$\delta_{12} = \delta_{21} = \Sigma \frac{n_1n_2L}{EA}$$

$$\delta_{22} = \Sigma \frac{n_2^2L}{EA}$$

The conditions of compatibility from which X_1 and X_2 can be determined are

$$\Delta_{10} + X_1\delta_{11} + X_2\delta_{12} = 0$$
$$\Delta_{20} + X_1\delta_{12} + X_2\delta_{22} = 0 \qquad [2.15]$$

EXAMPLE 2.2 *Find the reaction at B and the bar force in member BF, of the truss in Fig. 2.8. The cross-sectional area of the members in cm² are shown in parentheses. E is constant.*

The given truss is indeterminate to the second degree; it has one redundant member (internal indeterminacy) and one redundant reaction (external indeterminacy).

A base structure is obtained by removing the reaction at B and cutting the diagonal member BF. The two conditions of compatibility are:

$$\Delta_B + R_B\delta_{bb} + F_{BF}\delta_{bf} = 0$$
$$\Delta_F + R_B\delta_{bf} + F_{BF}\delta_{ff} = 0$$

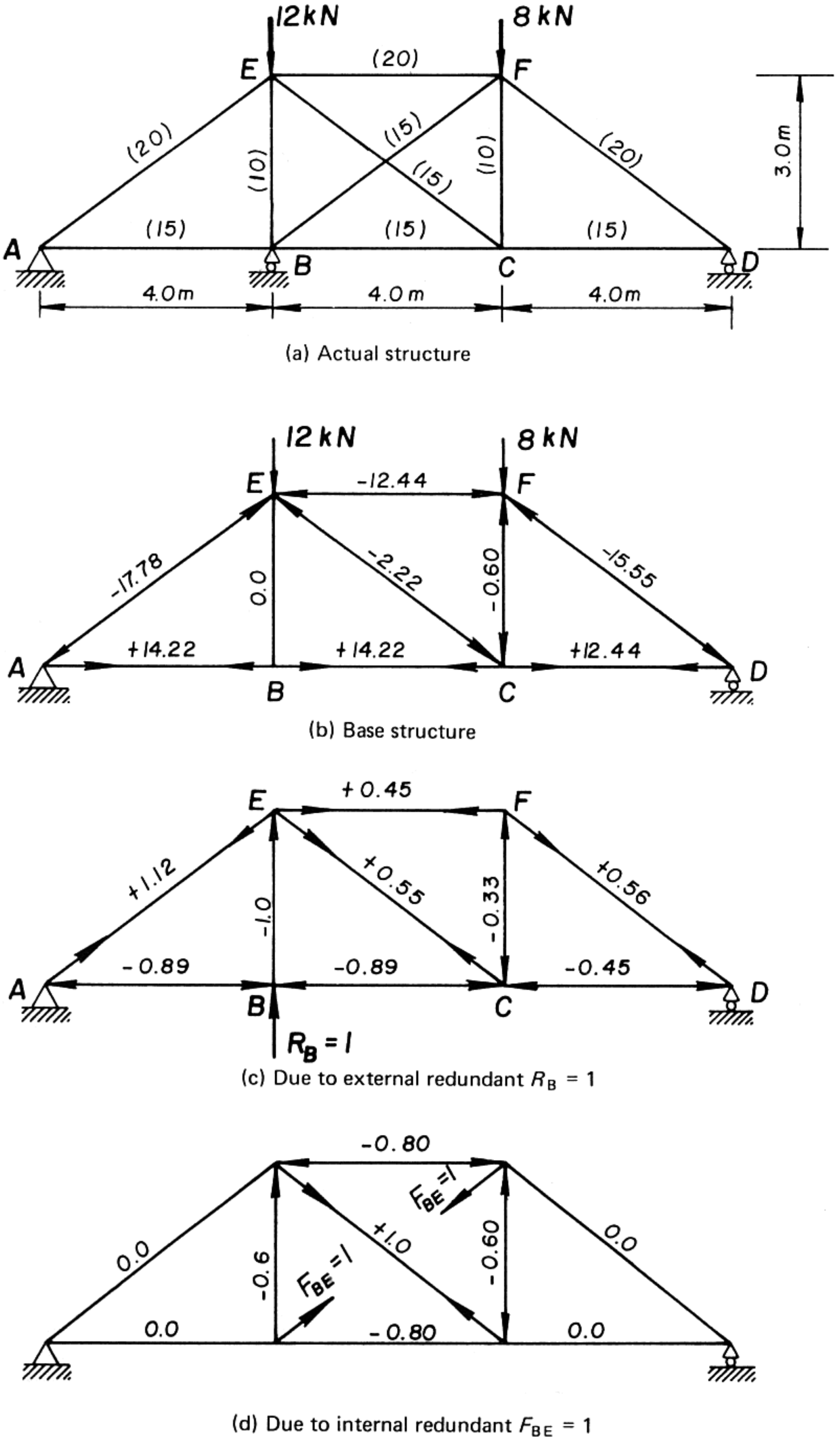
The displacements are computed in tabular form as shown in Table 2.1. Substituting the displacements:

$$-1680.8 + 134.91R_B + 54.52F_{BF} = 0$$
$$-202.3 + 54.52R_B + 118.1F_{BF} = 0$$

The solution of the simultaneous equations is

R_B = 14.44 tonnes (upward)

F_{BF} = –4.95 tonnes (compression)



(a) Actual structure

(b) Base structure

(c) Due to external redundant R_B = 1

(d) Due to internal redundant F_{BE} = 1

Figure 2.8

Table 2.1

Member	L cm	A cm^2	$\frac{L}{A}$	N	n_B ($R_B = 1$)	n_{BF} $F_{BF} = 1$	$\frac{Nn_BL}{A}$	$\frac{Nn_{BF}L}{A}$	$\frac{n_B^2L}{A}$	$\frac{n_{BF}^2L}{A}$	$\frac{n_Bn_{BF}L}{A}$
AE	500	20	25	17.78	+1.12	0	−497.84	0	+31.36	0	0
EF	400	20	20	−12.44	+0.45	−0.80	−111.96	+199.04	+ 4.05	+ 12.80	− 7.20
DF	500	20	25	−15.55	+0.55	0	−213.81	0	+ 7.56	0	0
BE	300	10	30	0	−1.00	−0.60	0	0	+30.0	+ 10.80	+18.0
CE	500	15	33.33	+ 2.22	+0.56	+1.00	− 41.44	− 73.99	+10.45	+ 33.30	+18.66
BF	500	15	33.33	0	0	1.00	0	0	0	+ 33.30	0
CF	300	10	30	− 1.33	−0.33	−0.60	+ 13.17	− 23.94	+ 3.27	+ 10.80	+ 5.94
AB	400	15	26.67	+14.22	−0.90	0	−339.81	0	+21.41	0	0
BC	400	15	26.67	+14.22	−0.90	−0.80	−339.81	−303.40	+21.41	+ 17.10	+19.12
CD	400	15	26.67	−12.44	−0.45	0	−149.3	0	+ 5.40	0	0
						Σ	−1680.8	−202.29	134.91	+118.1	54.52

2.4 ANALYSIS OF FRAMES

A framed structure is composed of an interconnected assemblage of beams and columns. A frame is said to be *rigid* if the members are rigidly connected. The basic analysis of statically indeterminate frames by the method of consistent deformation is essentially an extension of the same principle encountered in dealing with beams.

The members in frames are usually subjected to both axial and bending stresses; however, the axial stresses in the members of rigid frames are in most cases small compared with that of bending stresses. Thus, in computing the displacements in rigid frames for the conditions of consistent deformation, the effects of the axial stresses are usually neglected and the effects of bending stresses only are considered. This, however, does not mean that there are no axial forces in the members even if the change in the length of the members of rigid frames has insignificant effect on the values of the redundants.

To formulate the equations for the general case of multiply redundant structures, consider the frame shown in Fig. 2.9, which is triply statically indeterminate. Let the three support reactions at A be chosen as the redundants. When these redundants are removed, A will be displaced vertically and horizontally and will also rotate.

It will be seen that it will be convenient to adopt a slightly different notation with numerical subscripts for the redundants and displacements, which are defined as

$\Delta_{10}, \Delta_{20}, \Delta_{30}$ = displacements at A in the directions of X_1, X_2 and X_3 respectively, due to the applied loads on the base structure

$\delta_{11}, \delta_{21}, \delta_{31}$ = displacements at A on the base structure in the directions of X_1, X_2 and X_3 respectively, due to $X_1 = 1$ acting alone

$\delta_{12}, \delta_{22}, \delta_{32}$ = the above displacements on the base structure due to $X_2 = 1$ acting alone

$\delta_{13}, \delta_{23}, \delta_{33}$ = the above displacements on the base structure due to $X_3 = 1$ acting alone

If it is known that there are no support displacements, the equations of consistent deformation are

$$\begin{aligned}
&\Delta_{10} + X_1\delta_{11} + X_2\delta_{12} + X_3\delta_{13} = 0\\
&\Delta_{20} + X_2\delta_{12} + X_2\delta_{22} + X_3\delta_{23} = 0 \qquad [2.16]\\
&\Delta_{20} + X_3\delta_{13} + X_2\delta_{23} + X_3\delta_{33} = 0
\end{aligned}$$

The general equation for a structure with n redundants may then be written in

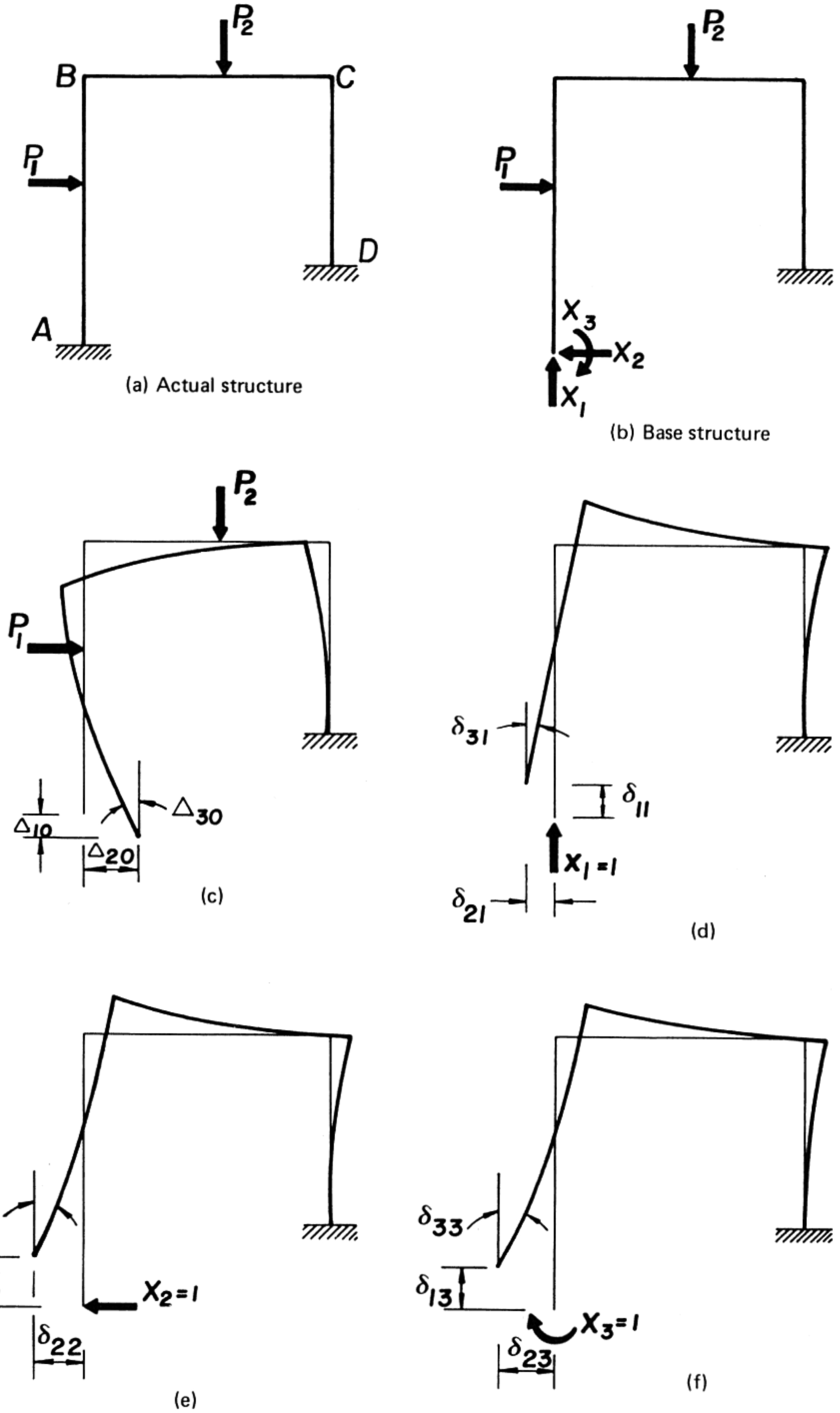

P2
B
C
P1
D
A
(a) Actual structure
P2
P1
X3
X2
X1
(b) Base structure
P2
P1
Δ10
Δ20
Δ30
(c)
δ31
δ11
X1 = 1
δ21
(d)
δ32
δ12
X2 = 1
δ22
(e)
δ33
δ13
X3 = 1
δ23
(f)

Figure 2.9

matrix form as

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \cdots \\ \Delta_{n0} \end{bmatrix} + \begin{bmatrix} \delta_{11} + \delta_{12} + \ldots + \delta_{1n} \\ \delta_{12} + \delta_{22} + \ldots + \delta_{2n} \\ \cdots \quad\quad \cdots \\ \delta_{1n} + \delta_{2n} + \ldots + \delta_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \qquad [2.17]$$

These equations, sometimes referred to as the elastic equations, form the basis for several different methods of analysing statically indeterminate structures. The coefficients $\delta_{11}, \delta_{12} \ldots$ of the redundants on the base structures, which are the displacements due to unit loads are known as *flexibility coefficients* or *influence coefficients*.

Whenever support displacements occur, the right-hand side of the equations may be suitably adjusted before solving the simultaneous equations.

In the general case where deflections occur as a consequence of flexural and axial deformation of members of the structure, displacements in the base structure due to the applied loads may be written in the form

$$\Delta_{10} = \int \frac{Mmdx}{EI} + \sum \frac{NnL}{EA} \qquad [2.18]$$

and the flexibility coefficients are

$$\delta_{ij} = \int \frac{m_i m_j dx}{EI} + \sum \frac{n_i n_j L}{EA} \qquad [2.19]$$

EXAMPLE 2.3 *Determine the reaction components at support D of the frame of Fig. 2.10.*

Since the horizontal and vertical displacements and the rotations at support D must be zero, the compatibility equations are

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The displacement coefficients are evaluated using the graphic multiplication method:

$$EI\Delta_{10} = \left(-\frac{3.0 \times 1.5}{2}\right)(2.5) + (-5.0 \times 6.0)(3) + \left(-\frac{3 \times 9}{2}\right)(3.0)$$

$$= -136.125$$

$$EI\Delta_{20} = \left(-\frac{3.0 \times 1.5}{2}\right)(-3) + (0.5 \times 6)(\pm 0.5) \mp \frac{(3 \times 9)}{2}(1.0)$$

$$= +8.25$$

$$EI\Delta_{30} = -\left(\frac{3.0 \times 1.5}{2}\right)(-1.0) + (-5 \times 6)(-1.0) + \left(\frac{3 \times 9}{2}\right)(-1.0)$$

$$= 45.75$$

$$EI\delta_{11} = \left(\frac{3 \times 1.5}{2}\right)(2.0) + (3.0)(5.0)(3.0)$$

$$= 49.5$$

$$EI\delta_{12} = \left(\frac{1.5 \times 3}{2}\right)(-3.0) + (1.5)(-0.5)$$

$$= -14.25 = EI\delta_{21}$$

$$EI\delta_{13} = \left(\frac{1.5 \times 3}{2}\right)(-1.0) + (1.5 \times 5)(-1.0)$$

$$= -17.25 = EI\delta_{31}$$

$$EI\delta_{22} = \left(-\frac{3 \times 3}{2}\right)(-2.0) + \left(-\frac{1.5 \times 3}{2}\right)(-3.0) + \left(-\frac{3 \times 3}{2}\right)(-2.0)$$

$$+ \left(\frac{2 \times 2}{2}\right)\frac{4}{5}$$

$$= 34.17$$

$$EI\delta_{23} = \left(-\frac{3 \times 3}{2}\right)(-1.0) + \left(-\frac{1.5 \times 3}{2}\right)(-1.0) + \left(-\frac{3 \times 3}{2}\right)(-1.0)$$

$$+ \left(\frac{2 \times 2}{2}\right)(-1.0)$$

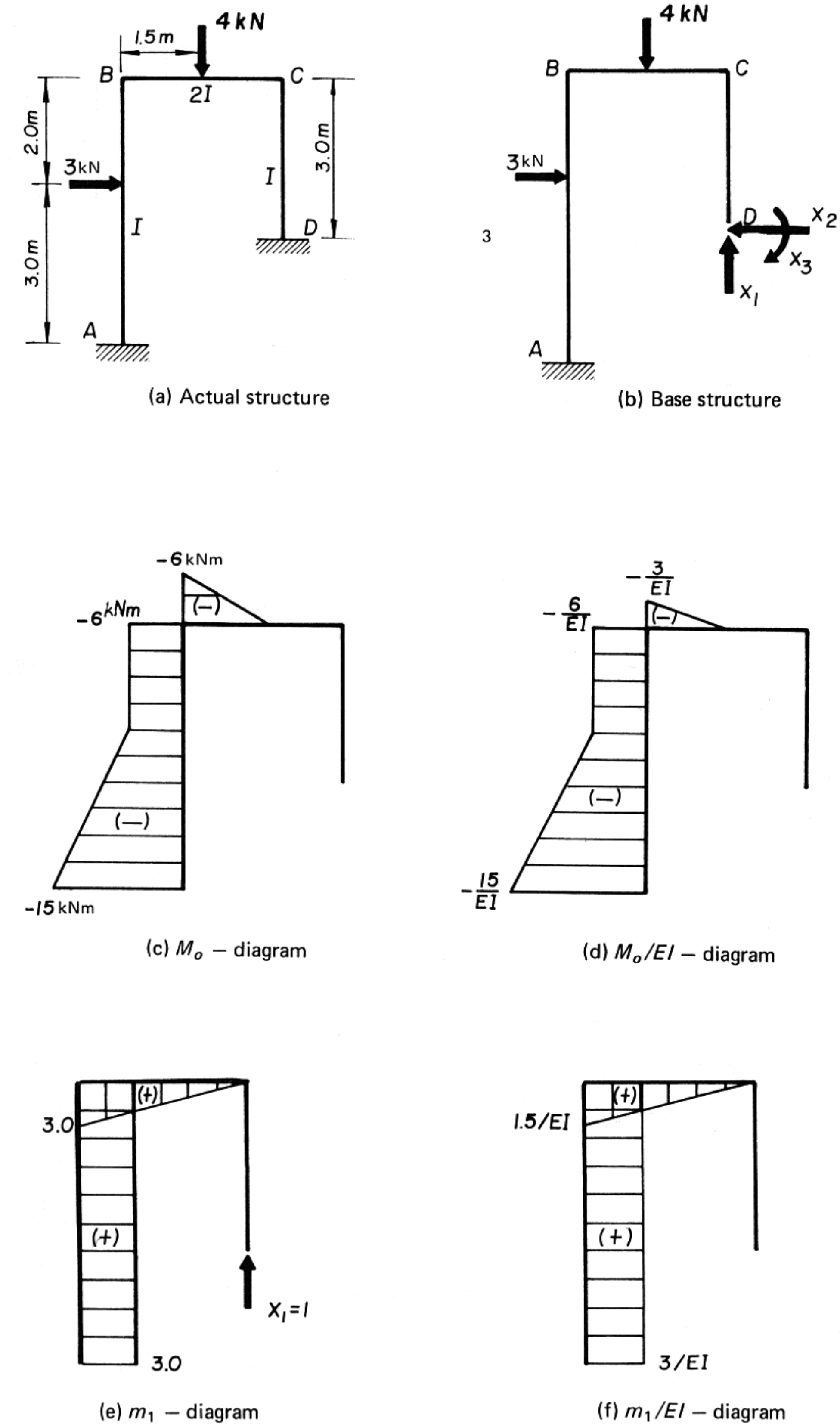
$$= 11.5 = EI\delta_{32}$$

$$EI\delta_{33} = (-1.0 \times 3)(-1.0) + (-1.0 \times 3)(-1.0) + (-1.0 \times 5)(-1.0)$$

$$= 9.5$$

Substituting these values into the elastic equations:

$$\begin{bmatrix} -136.125 \\ 8.25 \\ 45.75 \end{bmatrix} + \begin{bmatrix} 49.5 & -14.25 & -17.25 \\ -14.25 & 34.17 & 11.5 \\ -17.25 & 11.5 & 9.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



(a) Actual structure

(b) Base structure

(c) M_o – diagram

(d) M_o/EI – diagram

(e) m_1 – diagram

(f) m_1/EI – diagram

Figure 2.10

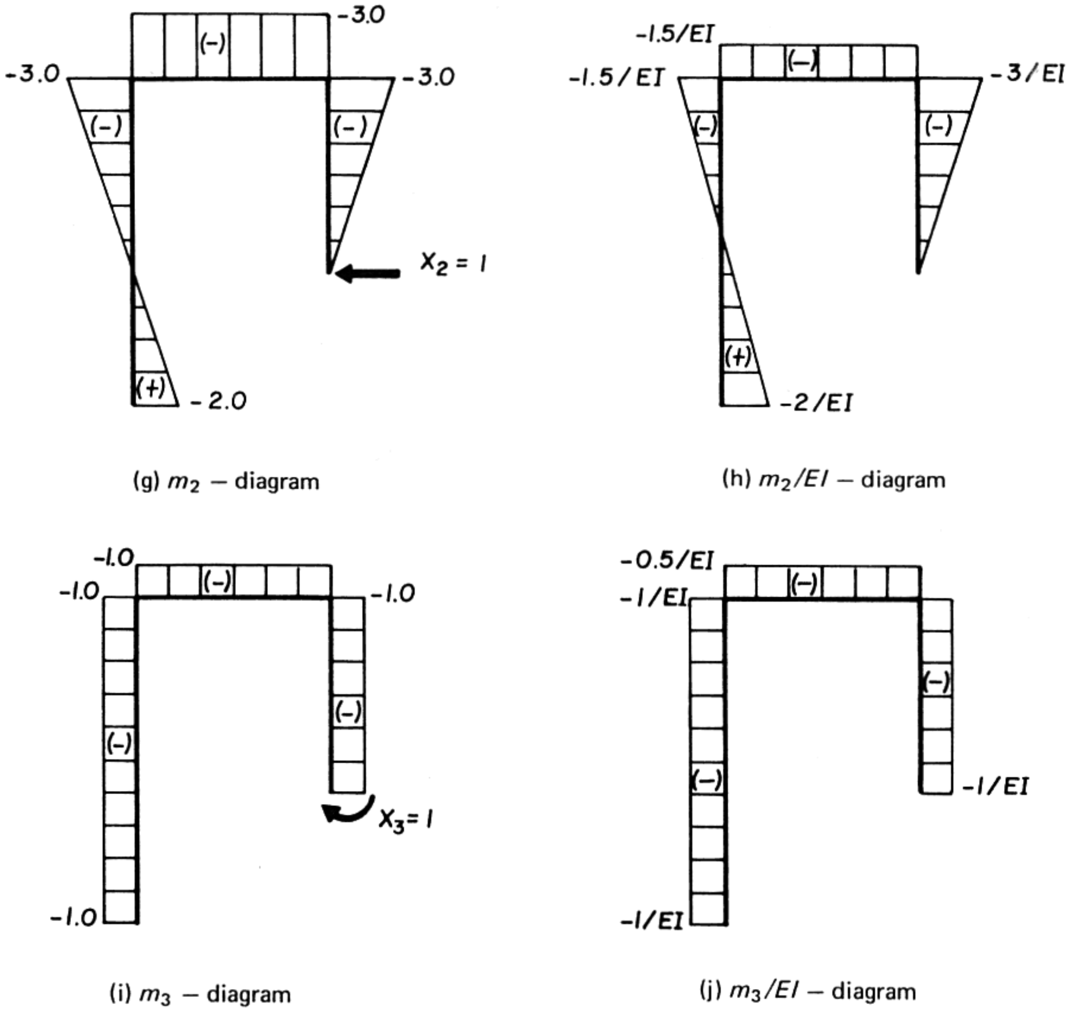


Figure 2.10 contd.

Solving the simultaneous equation:

$V_D = X_1 = 2.34$ kN (upward)

$H_D = X_2 = 1.56$ kN (to left)

$M_D = X_3 = 2.45$ kN m (counter-clockwise)

2.5 THE ELASTIC CENTRE METHOD

The elastic centre method is a special method of solving statically indeterminate structures of the one-loop form. Rigid-jointed portal frames, single-bay gabled bents, single-span arch, closed or ring structures are examples of the type of problems easily solved by this method.

Consider a fixed arch as in Fig. 2.11(a) under an arbitrary loading which produces bending moments M_0 in the primary structure.

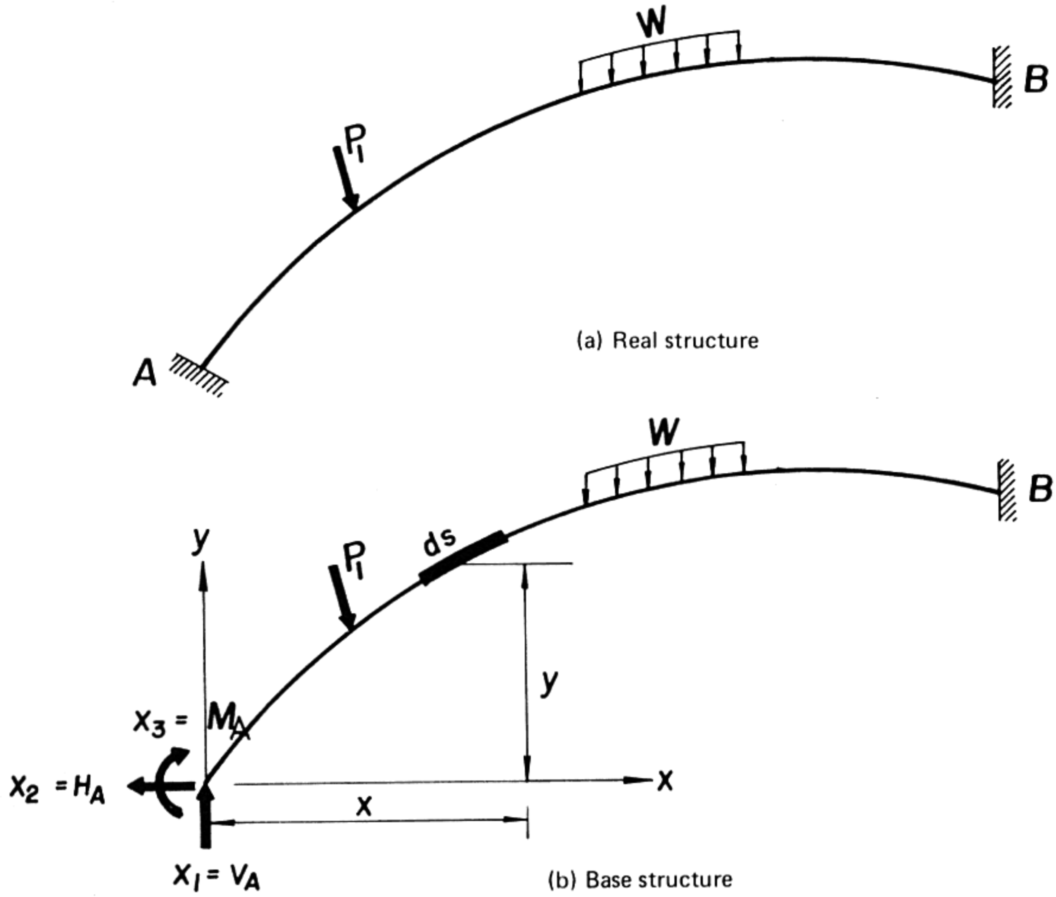


Figure 2.11

The primary structure is taken to be a cantilever as in Fig. 2.11(b), where the left support is removed and the redundant reactions $X_1 = V_A$, $X_2 = H_A$ and $X_3 = M_A$ are applied at the support point. The compatibility equations are

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{21} & \delta_{31} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{23} & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [2.20]$$

The displacements are evaluated as

$$\Delta_{10} = \int \frac{M_o m_1 ds}{EI} = \int \frac{M_o x ds}{EI}$$

$$\Delta_{20} = \int \frac{M_o m_2 ds}{EI} = \int \frac{M_o y ds}{EI} \qquad [2.21]$$

$$\Delta_{30} = \int \frac{M_o m_3 ds}{EI} = \int \frac{M_o ds}{EI}$$

since $m_1 = x$, $m_2 = y$ and $m_3 = 1$.

Also

$$\delta_{11} = \int \frac{m_1^2 ds}{EI} = \int \frac{x^2 ds}{EI}$$

$$\delta_{22} = \int \frac{m_2^2 ds}{EI} = \int \frac{y^2 ds}{EI}$$

$$\delta_{33} = \int \frac{m_3^2 ds}{EI} = \int \frac{ds}{EI} \qquad [2.22]$$

$$\delta_{12} = \delta_{21} = \int \frac{m_1 m_2 ds}{EI} = \int \frac{xyds}{EI}$$

$$\delta_{13} = \delta_{31} = \int \frac{m_1 m_3 ds}{EI} = \int \frac{xds}{EI}$$

$$\delta_{23} = \delta_{32} = \int \frac{m_2 m_3 ds}{EI} = \int \frac{yds}{EI}$$

If ds/EI is considered as an elemental area of length ds and a width normal to the arch axis of $1/EI$, then the following interpretations may be made:

δ_{11} and δ_{22} = moment of inertia of $1/EI$ area about the y axis and x axis, respectively

δ_{33} = total $1/EI$ area of the arch

δ_{12} = product of inertia of $1/EI$ about the given axis

δ_{23} = statical moment of $1/EI$ area about the y axis and x axis, respectively

If the origin of the axes can be transferred to the centroid or *elastic centre* of the *elastic area*, ds/EI, the computation may be simplified by the virtue of the fact that δ_{13} and δ_{23}, being the statical moments of elastic areas, disappear. Also if the axes through the elastic centre are the *principal axes*, δ_{12}, being the product of inertia, also vanishes.

It is statically possible to transfer the forces $X_1 = V_A$ and $X_2 = H_A$ to any point, provided $X_3 = M_A$ is properly modified, since any force may be replaced by an equal parallel force acting through any arbitrarily chosen point and a couple. Accordingly, the redundants may be applied at point 0 (x_0, y_0) which is attached to A by a perfectly rigid arm (Fig. 2.12). It is clear that this arm does not fundamentally change the structure, since, being rigid, it makes no direct contribution to the deflection of the arch.

Taking 0 as the origin of coordinates,

$$V_A = X_1$$

$$H_A = X_2 \qquad [2.23]$$

$$M_A = X_3 + X_1 x_0 + X_2 y_0$$

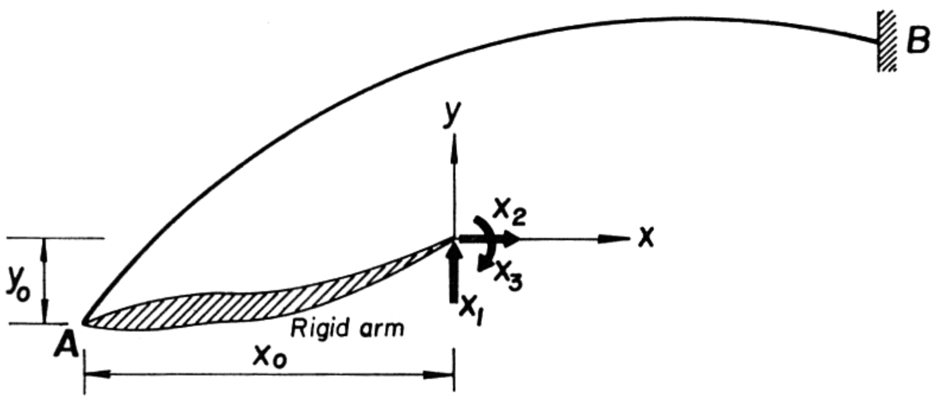


Figure 2.12

If the structure is symmetrical, the centroidal-principal axis will coincide with the axis of symmetry, and hence the product of inertia will be zero. Thus

$$\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$$

The corresponding compatibility equations are

$$\begin{aligned} \Delta_{10} + X_1\delta_{11} &= 0 \\ \Delta_{20} + X_2\delta_{22} &= 0 \\ \Delta_{30} + X_3\delta_{33} &= 0 \end{aligned} \qquad [2.24]$$

where the redundant reactions are taken at the elastic centre.

If the structure is not symmetrical where the x and y axes are not the principal axes, δ_{12} will not vanish, while δ_{13} and δ_{23} disappear. The compatibility equations for this case are

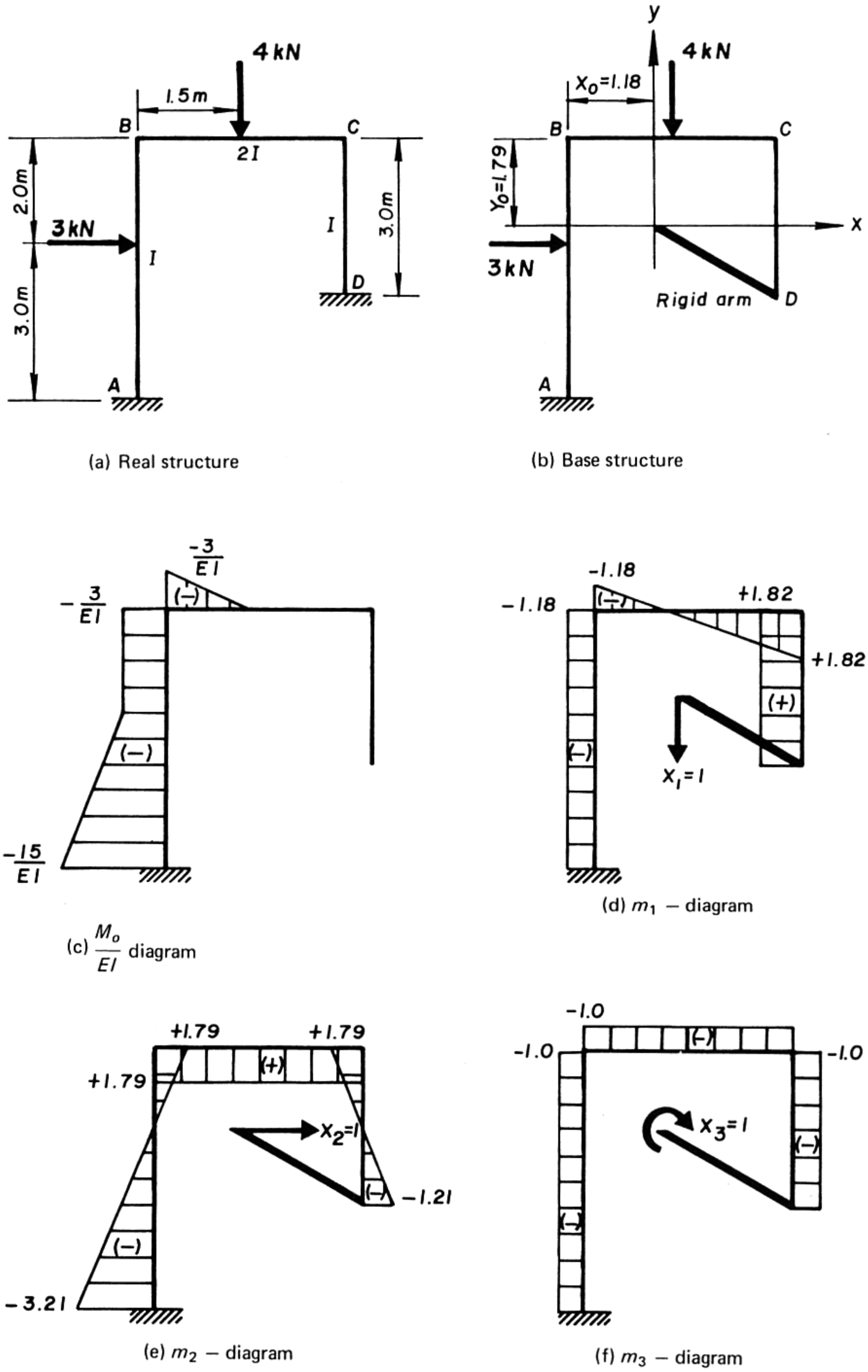
$$\begin{aligned} \Delta_{10} + X_1\delta_{11} + X_2\delta_{21} &= 0 \\ \Delta_{20} + X_1\delta_{12} + X_2\delta_{22} &= 0 \\ \Delta_{30} + X_3\delta_{33} &= 0 \end{aligned} \qquad [2.25]$$

Solving the simultaneous equations

$$\begin{aligned} X_1 &= -\frac{\Delta_{10} - (\delta_{12}/\delta_{22})\Delta_{20}}{\delta_{11} - (\delta_{12}/\delta_{22})\delta_{12}} \\ X_2 &= -\frac{\Delta_{20} - (\delta_{12}/\delta_{11})\Delta_{10}}{\delta_{22} - (\delta_{12}/\delta_{11})\delta_{12}} \\ X_3 &= -\Delta_{30}/\delta_{33} \end{aligned} \qquad [2.26]$$

EXAMPLE 2.4 *Determine the reaction components at D of the rigid frame of Fig. 2.13, using the elastic centre method.*

The elastic centre is located by taking moments about AB for the x coordinate



4 kN
1.5 m
B
C
2I
2.0m
3 kN
I
I
3.0m
D
3.0m
A
(a) Real structure
y
4 kN
X_0=1.18
B
C
Y_0=1.79
X
3 kN
Rigid arm
D
A
(b) Base structure
-3/EI
(–)
– 3/EI
(–)
–15/EI
(c) M_o/EI diagram
-1.18
–1.18
+1.82
+1.82
(+)
(–)
X_1=1
(d) m_1 – diagram
+1.79
+1.79
+1.79
(+)
X_2=1
(–)
–1.21
(–)
–3.21
(e) m_2 – diagram
-1.0
-1.0
-1.0
(–)
X_3=1
(–)
(–)
(f) m_3 – diagram

Figure 2.13

and about BC for the y coordinate:

$$x_o = \frac{\frac{(3)(3)}{1} + \frac{(3)(1.5)}{2}}{5 + \frac{3}{2} + 3} = 1.18 \text{ m}$$

$$y_o = \frac{\frac{(5)(2.5)}{1} + \frac{(3)(3)}{2}}{9.5} = 1.79 \text{ m}$$

The displacement coefficients are computed using graphic multiplication method:

$$EI\Delta_{10} = \left(\frac{-3 \times 1.5}{2}\right)(-0.68) + (-6 \times 5)(-1.18) + \left(-\frac{9 \times 3}{2}\right)(-1.18)$$

$$= 52.87$$

$$EI\Delta_{20} = \left(-\frac{3 \times 1.5}{2}\right)(1.79) + (-6 \times 5)\left(\frac{-3.21 + 1.79}{2}\right) + \left(-\frac{9 \times 3}{2}\right)(-2.2$$

$$= 47.11$$

$$EI\Delta_{30} = \left(-\frac{3 \times 1.5}{2}\right)(-1.0) + (-6 \times 5)(-1) + \left(-\frac{9 \times 3}{2}\right)(-1.0)$$

$$= 47.75$$

$$EI\delta_{11} = (-1.18 \times 5)(-1.18) + \frac{1}{2}\left(-\frac{1.18 \times 1.8}{2}\right)\left(-\frac{2}{3} \times 1.18\right)$$

$$+ \frac{1}{2}\left(-\frac{1.82 \times 1.8}{2}\right)\left(-\frac{2}{3} \times 1.18\right) + (3 \times 1.82)(1.82)$$

$$= 18.18$$

$$EI\delta_{22} = \left(-\frac{3.21 \times 3.21}{2}\right)\left(-\frac{2}{3} \times 3.21\right) + \left(\frac{1.79 \times 1.79}{2}\right)\left(\frac{2}{3} \times 1.79\right)$$

$$+ \frac{1}{2}(1.79 \times 3)(1.79) + \left(-\frac{1.79 \times 1.79}{2}\right)\left(-\frac{2}{3} \times 1.79\right)$$

$$+ \left(\frac{1.21 \times 1.2}{2}\right)\left(\frac{2}{3} \times 1.21\right)$$

$$= 20.25$$

$$EI\delta_{33} = (5 \times 1)(1) + \frac{1}{2}(3 \times 1)(1) + (3 \times 1)(1)$$

$$= 9.5$$

Since the axes through 0 are not principal axes, δ_{12} will not vanish. Thus,

$$EI\delta_{12} = -\frac{(3.21 + 1.79)}{2} \times 5\,(-1.18) + \frac{1}{2}\,\frac{(1.82 - 1.18)}{2} \times 3\,(1.79)$$

$$+ \frac{1.79 - 1.21}{2} \times 3\,(1.82)$$

$$= 6.63$$

The elastic equations are

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & 0 \\ \delta_{12} & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

substituting

$$\begin{bmatrix} 52.87 \\ 47.11 \\ 47.75 \end{bmatrix} + \begin{bmatrix} 18.18 & 6.63 & 0 \\ 6.63 & 20.25 & 0 \\ 0 & 0 & 9.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of the simultaneous equations is

$X_1 = -2.34$ kN

$X_2 = -1.56$ kN

$X_3 = -4.82$ kN m

The values of the redundants at support D are:

$V_D = 2.34$ kN (upward)

$H_D = 1.56$ kN (left)

$M_D = 4.82 - 2.34 \times 1.82 + 1.56 \times 1.21$

$= 2.45$ kN m (counter-clockwise)

2.6 THE THREE-MOMENT EQUATIONS

Consider a continuous beam with n spans as shown in Fig. 2.14. This beam is indeterminate to the $(n - 1)$th degree when the support reactions are taken as the redundants, each of which contain all the unknowns. However, when support moments are used as the redundants, although the same number of equations must eventually be solved, each equation contains only three of the unknowns. The latter choice of redundants localises the loading conditions on

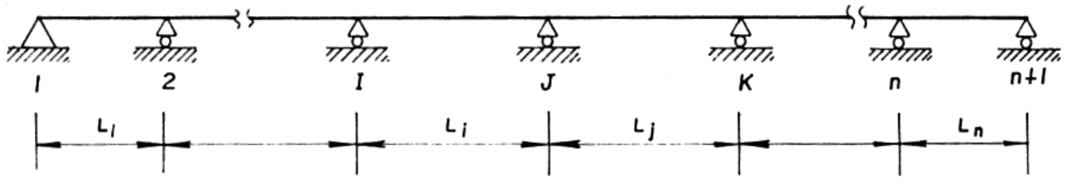


Figure 2.14

the base structure and the resulting relationship between the redundants permits the equations to be written in a simple and systematic manner. These equations express the three-moment equations first presented by the French engineer Clapeyron.

The three-moment equation expresses the relation between the bending moments at the three successive supports of a continuous beam. The relation is derived from the continuity of the elastic curve when the compatibility equations are obtained in terms of the support slopes of adjacent spans.

Consider two adjacent spans IJ and JK of a continuous beam shown in Fig. 2.15. The moment of inertia is considered constant between I and J and equal to I_i and likewise constant between J and K and equal to I_j. The beam is assumed to be initially straight, and the support settlements amounting to Δ_i, Δ_j and Δ_k take place at support I, J and K respectively, as indicated by the heavy line in Fig. 2.15(e).

Compatibility equations are written at each support expressing the equality of end slopes at adjacent spans. The condition of continuity of the slope gives

$$\frac{\Delta_j - \Delta_i + \delta_i}{L_i} = \frac{\Delta_k - \Delta_j - \delta_k}{L_j}$$

Rearrange the equation as

$$\frac{\delta_i}{L_i} + \frac{\delta_k}{L_j} = \frac{\Delta_i - \Delta_j}{L_i} + \frac{\Delta_k - \Delta_j}{L_i} \qquad [2.27]$$

But from the Second Theorem of the Area-moment Method,

$$\delta_i = \frac{1}{E_i I_i}\left[A_i\bar{x}_i + \left(\frac{1}{2}M_i L_i\right)(L_i/3) + \left(\frac{1}{2}M_j L_i\right)(2L_i/3)\right]$$

$$\delta_k = \frac{1}{E_j I_j}\left[A_i\bar{x}_j + \left(\frac{1}{2}M_k L_j\right)(L_j/3) + \left(\frac{1}{2}M_j L_j\right)(2L_j/3)\right] \qquad [2.28]$$

Combining [2.27] and [2.28] gives Clapeyron's Equation of Three Moments:

$$M_i\left(\frac{L_i}{E_i I_i}\right) + 2M_j\left(\frac{L_i}{E_i I_i} + \frac{L_j}{E_j I_j}\right) + M_k\left(\frac{L_j}{E_j I_j}\right)$$

$$+ 6\left(\frac{A_i\bar{x}_i}{E_i I_i L_i} + \frac{A_j\bar{x}_i}{E_j I_j L_j}\right) = 6\left(\frac{\Delta_i - \Delta_j}{L_i} + \frac{\Delta_k - \Delta_j}{L_j}\right) \qquad [2.29]$$

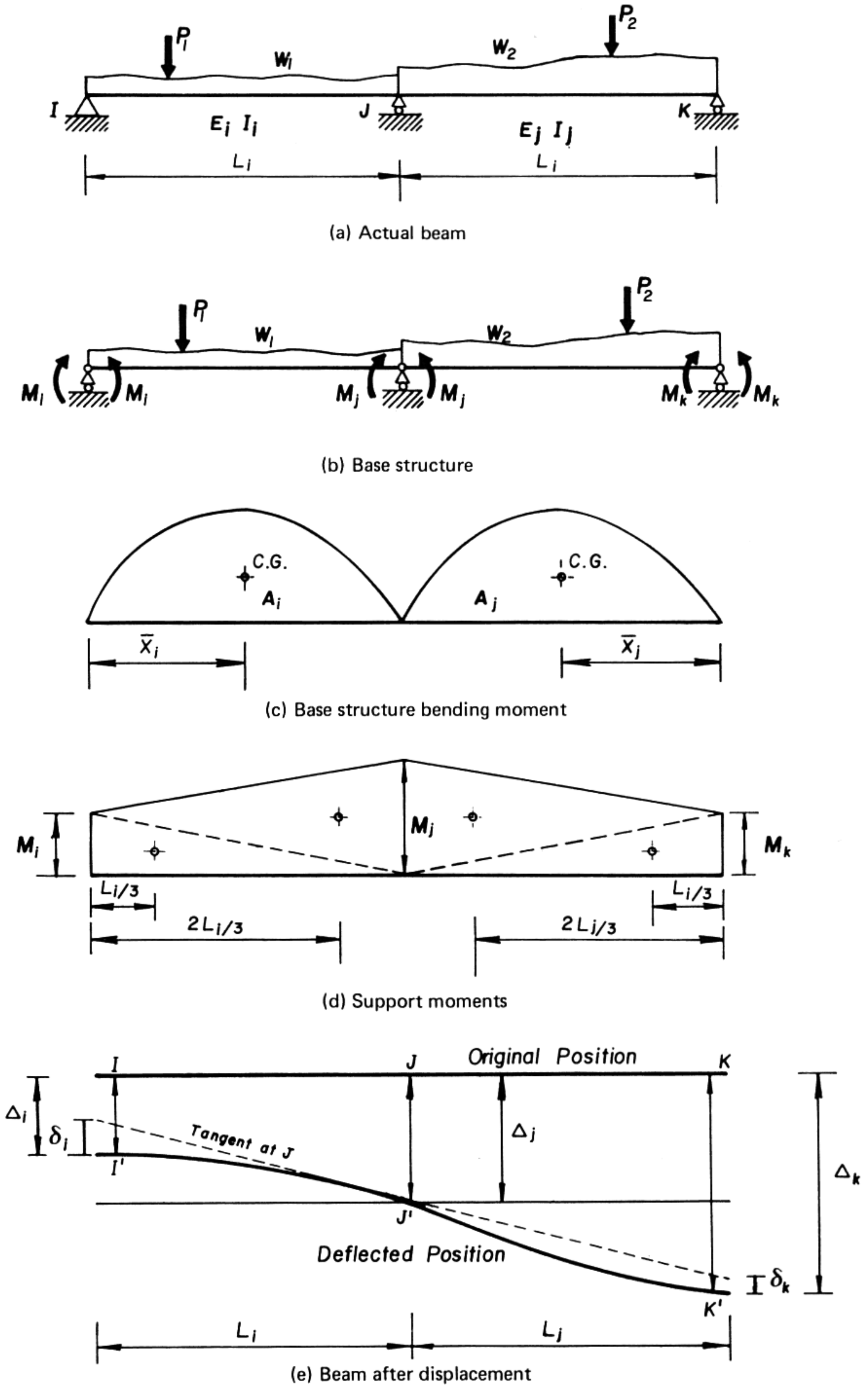

P1
W1
P2
W2
I
J
K
Ei Ii
Ej Ij
Li
Li
(a) Actual beam
Mi
Mj
Mk
(b) Base structure
C.G.
Ai
Aj
X̄i
X̄j
(c) Base structure bending moment
Mi
Mj
Mk
Li/3
2Li/3
Li/3
2Lj/3
(d) Support moments
Original Position
Tangent at J
I'
J'
K'
Deflected Position
Δi
δi
Δj
Δk
δk
Li
Lj
(e) Beam after displacement

Figure 2.15