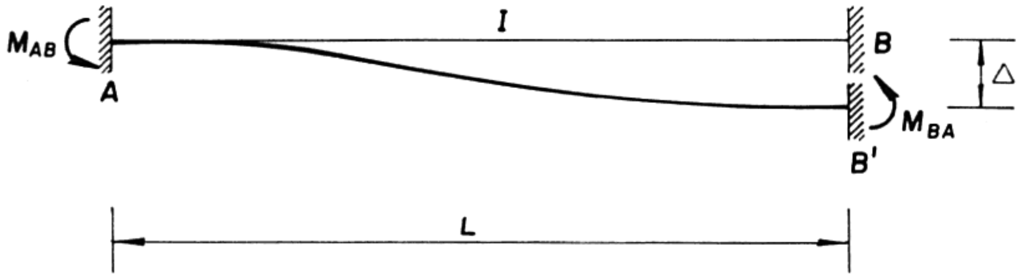


Figure 4.5

displacement of supports can be considered by referring to Fig. 4.5. The beam AB is fixed against rotation at A and B. The end B is deflected downward an amount Δ while both ends are restrained against rotation.

The slope deflection equation is used to determine the value of M_{AB}.

$$M_{AB} = M_{BA} = -2EK(-3\Delta/L)$$

$$= \frac{6EK\Delta}{L}$$

The corresponding *fixed-end* moments are

$$M^F_{AB} = M^F_{BA} = \frac{6EI\Delta}{L^2} \qquad [4.5]$$

Similarly, the fixed-end moments due to rotation θ_A at A are

$$M^F_{AB} = 2M^F_{BA} = \frac{4EI\theta_A}{L} \qquad [4.6]$$

A beam with one end hinged which deflects Δ is shown in Fig. 4.6.

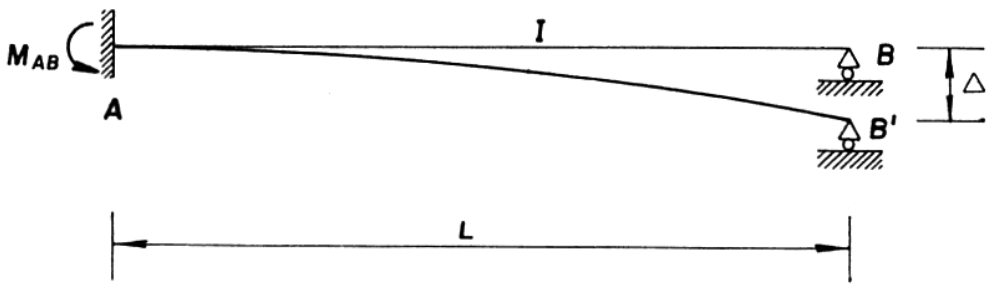


Figure 4.6

The slope deflection equation gives

$$M_{BA} = -2EK(2\theta_B - 3\Delta/L) = 0$$

$$\theta_B = 3\Delta/2L$$

Thus

$$M_{AB} = -2EK\left(\theta_B - \frac{3\Delta}{L}\right)$$

$$= \frac{3EK\Delta}{L^2}$$

The corresponding *fixed-end* moment is

$$M^F_{AB} = \frac{3EI\Delta}{L^2}$$

$$M_{BA} = 0 \qquad [4.7]$$

The appropriate fixed-end moment corresponding to the relative displacements will then be introduced into the moment distribution scheme. These moments are treated in exactly the same manner as those due to the applied loadings.

4.6 MOMENT DISTRIBUTION METHOD FOR FRAME ANALYSIS

In applying the moment distribution method to frame problems, there is no inherent limitation which prevents the use of the basic concepts developed so far. The method can be applied to frame problems with some changes in the book keeping scheme. For frames with fewer number of joints, the tabular scheme of calculations may be more convenient to adopt. However, for frames with numerous joints the sketch of the structure itself may be more practical to use. One system of recording the moments is to place the computed value *under* the beam on the *left* and *over* the beam on the *right*. Similarly, the column moments are recorded to the *left* on the *top* of the column and to the *right* on the *bottom*. This arrangement of showing the calculations on the structure itself is shown in Fig. 4.7.

Frame problems are classified in two categories:

(a) Frames without sidesway
(b) Frames with sidesway

4.6.1 Frames Without Sidesway

Frames without sidesway are analysed in the same manner as continuous beams. In the case of frames there are frequently more than two members meeting at a joint so that the joint moments distribute among all members according to the appropriate distribution factors. Consequently, for such frames there is a need of a more systematic arrangement of computations.

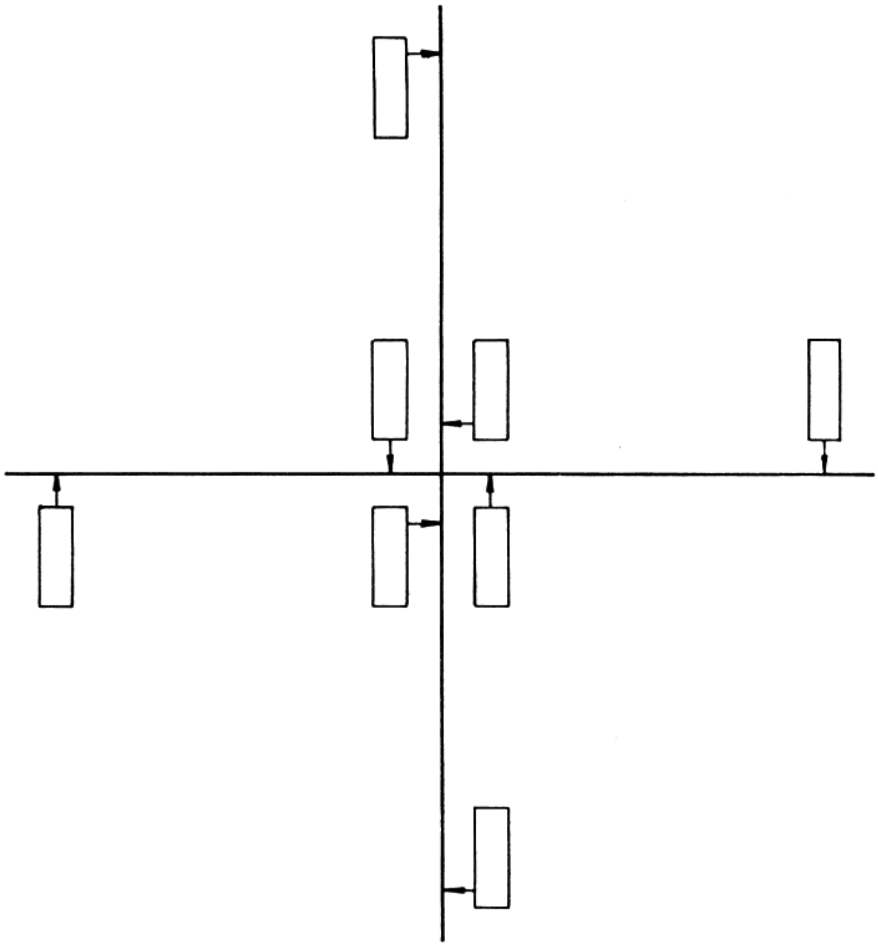


Figure 4.7

EXAMPLE 4.3 *Determine the joint moments of the frame shown in Fig. 4.8.*

Relative Stiffnesses and Distribution Factors

$$K_{AB} = \frac{12}{3} = 4$$

$$K_{BC} = \frac{2 \times 12}{6} = 4$$

$$K_{CD} = \frac{12}{3} = 4$$

$$K_{CE} = \frac{3}{4} \times \frac{1.5 \times 12}{4} = 3.375$$

$$(DF)_{BA} = \frac{4}{4 + 4} = 0.5$$

$$(DF)_{BC} = \frac{4}{4+4} = 0.5$$

$$(DF)_{CB} = \frac{4}{4+4+3.375} = 0.352$$

$$(DF)_{CD} = \frac{4}{11.375} = 0.352$$

$$(DF)_{CE} = \frac{3.375}{11.375} = 0.296$$

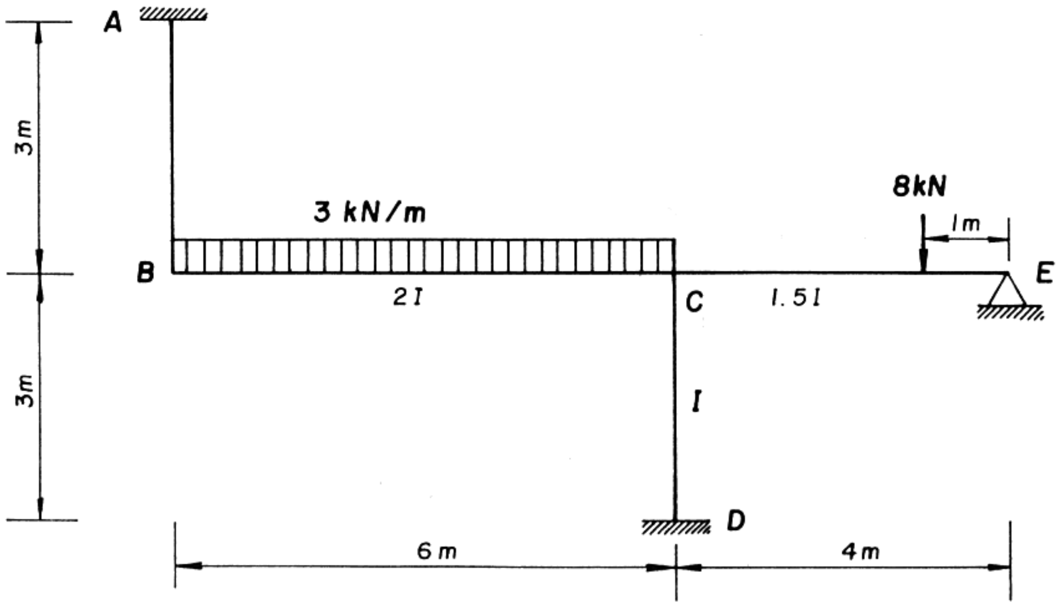


Figure 4.8

Fixed-End Moments

$$M^F_{BC} = -M^F_{CB} = \frac{3(6)^2}{12} = 9 \text{ kN m}$$

$$M^F_{CE} = \frac{8(3)(1)^2}{4^2} = 1.5 \text{ kN m}$$

$$M^F_{EC} = \frac{8(1)(3)^2}{16} = -4.5 \text{ kN m}$$

The moment distribution is performed in tabular form since the number of joints is few.

Table 4.6

Joint	A	B		C			D	E
Member	AB	BA	BC	CB	CE	CD	DC	EC
K	4	4	4	4	3.375	4	4	3.375
DF	0	0.5	0.5	0.352	0.296	0.352	0	
Fixed-end moment			+9.00	−9.00	+1.50 +2.25			−4.50 +4.50
			+0.92	+1.85	+1.55	+1.85	+0.93	
	−2.48	−4.96	−4.96	−2.48				
			+0.44	+0.87	+0.74	+0.87	+0.44	
	−0.11	−0.22	−0.11					
			+0.02	+0.04	+0.03	+0.04	+0.02	
Total	−2.59	−5.19	+5.19	−8.83	+6.07	+2.76	+1.39	0.0

4.6.2 Frames With Sidesway

Rectangular Frames

When a frame is loaded laterally, or in the case when the loading or the frame itself is unsymmetrical, sidesway or joint translations occur. As the joint displacements are unknown the fixed-end moments due to the displacements cannot be calculated. The solution for frames with a single mode of sway is indicated in Fig. 4.9. In applying the moment distribution method the joints are first assumed to be held against sidesway by introducing ***artificial restraint*** R, at the appropriate joints. The fixed-end moments caused by the applied loads are then distributed to obtain the non-sway balanced end moments. Next, the magnitude of the reaction R, at the ***artificial restraint*** is determined from the considerations of equilibrium of the members. The effect of the restraint R is determined by permitting an arbitrary sway to take place, unaccompanied initially by rotation at the joints. These fixed-end moments are then distributed and subsequently, the force F necessary to maintain the frame in its swayed position is determined from the equilibrium condition of the column shears. The actual magnitude of lateral force, F, consistent with the condition necessary to eliminate the ***artificial restraint*** R, and therefore the moments caused by the sidesway, are determined from the condition:

$$R = kF \qquad [4.8]$$

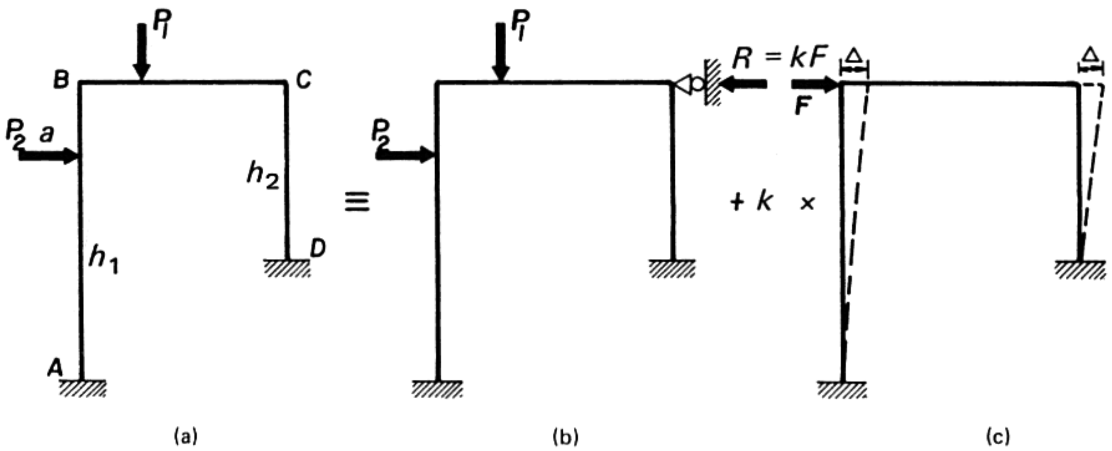


Figure 4.9

The final set of end moments are obtained by adding the original set of moments to the sway moments multiplied by the *correction factor k*. This process is illustrated by the frame shown in Fig. 4.9.

Note that the frame shown in Fig. 4.9(a) is equivalent to the sum of Fig. 4.9(b) and Fig. 4.9(c). In Fig. 4.9(b) the frame is prevented from sidesway by the artificial restraint R applied at joint C. The arbitrary displacement Δ occurs at C' with the joints held against rotation and when the fixed-end moments are distributed the horizontal force that caused the sidesway F is determined from equilibrium conditions. The shear condition of the given frame is

$$H'_A + H'_D - P_2 = R$$

But

$$H'_A = \frac{M'_{AB} + M'_{BA}}{h_1} + \frac{P_2 a}{h_1}$$

$$H'_D = \frac{M_{CD} + M_{DC}}{h_2}$$

Thus

$$\frac{M'_{AB} + M'_{BA}}{h_1} + \frac{P_2 a}{h_1} + \frac{M'_{CD} + M'_{DC}}{h_2} - P_2 = R$$

Similarly

$$H''_A + H''_D = F$$

or

$$\frac{M''_{AB} + M''_{BA}}{h_1} + \frac{M''_{CD} + M''_{DC}}{h_2} = F \qquad [4.9]$$

But

$$R = kF \qquad [4.10]$$

Therefore by superposition condition

$$M_{AB} = M'_{AB} + kM''_{AB}$$
$$M_{BA} = M'_{BA} + kM''_{BA}\text{, etc.} \qquad [4.11]$$

EXAMPLE 4.4 *Determine the joint moments of the frame shown in Fig. 4.10.*

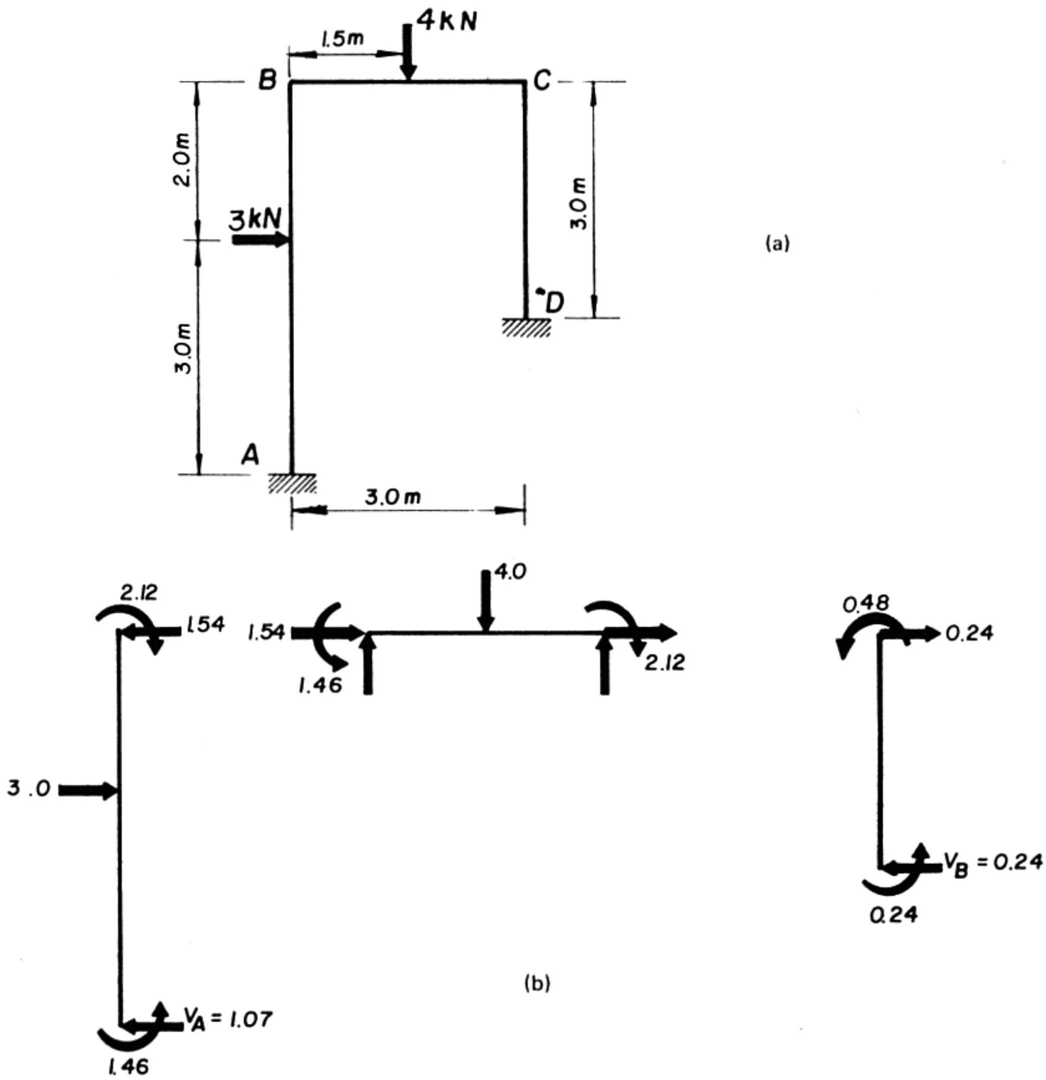


Figure 4.10

Relative Stiffnesses and Distribution Factors

$$K_{AB} = \frac{15}{5} = 3$$

$$K_{BC} = \frac{15 \times 2}{3} = 10$$

$$K_{CD} = \frac{15}{3} = 5$$

$$(DF)_{BA} = \frac{3}{10+3} = 0.231$$

$$(DF)_{BC} = \frac{10}{13} = 0.769$$

$$(DF)_{CB} = \frac{10}{10+5} = 0.667$$

$$(DF)_{CD} = \frac{5}{15} = 0.333$$

Fixed-End Moments

$$M_{AB}^{F} = \frac{3(3)(2)^2}{3^2} = 1.44 \text{ kN m}$$

$$M_{BA}^{F} = -\frac{3(2)(3)^2}{25} = -2.16 \text{ kN m}$$

$$M_{CB}^{F} = -M_{BC}^{F} = \frac{4(1.5)(1.5)^2}{9} = 1.50 \text{ kN m}$$

(a) Moment Distribution without Sidesway

Table 4.7 Distribution without Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	3	3	10	10	5	5
DF	0.0	0.231	0.769	0.667	0.333	0.0
Fixed-end moment	+1.44	−2.16	+1.5	−1.5		
			+0.5	+1.0	+0.50	+0.25
	+0.02	+0.04	+0.12	+0.06		
				−0.04	−0.02	−0.01
Total	+1.46	−2.12	+2.12	−0.48	+0.48	+0.24

By using the end moments on the free-body diagrams as shown in Fig. 4.10(a) the end shears are determined to be

$$V_A = \frac{3 \times 2 + 1.46 - 2.12}{5} = 1.068 \text{ kN} \quad \text{(left)}$$

$$V_B = \frac{0.48 + 0.24}{3} = 0.240 \text{ kN} \quad \text{(left)}$$

The artificial joint restraint is

$$R = 3.0 - 1.068 - 0.24 = 1.692 \text{ kN} \quad \text{(left)}$$

(b) Moment Distribution with Sidesway

Assume the frame to sway an arbitrary amount Δ such that the fixed-end moments in the columns are

$$M^F_{AB} = -M^F_{CD} = \frac{6EI\Delta}{L^2}$$

$$= \frac{6EI\Delta}{5^2} = 0.24EI$$

$$M^F_{CD} = +M^F_{DC} = \frac{6EI\Delta}{3^2} = 0.667EI$$

Taking $EI = 10$, distribution of the fixed-end moments due to sidesway is shown in Table 4.8.

Table 4.8 Distribution with Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0.0	0.231	0.769	0.667	0.333	0.0
Fixed-end moment	+2.4	+2.4			+6.67	+6.67
	−0.27	−0.55	−1.85	−0.93		
			−1.92	−3.83	−1.91	−0.95
		+0.44	+1.48	+0.74		
			−0.24	−0.49	−0.25	−0.12
	+0.03	+0.06	+0.18			
			+0.03	−0.06	−0.03	
Total	+2.38	+2.36	−2.36	−4.48	+4.48	+5.59

The column shears are

$$V_{AB} = \frac{2.38 + 2.36}{5.0} = 0.948 \text{ kN}$$

$$V_{DC} = \frac{4.48 + 5.59}{3.0} = 3.356 \text{ kN}$$

The net shear is

$$F = V_{AB} + V_{DC}$$

$$= 0.948 + 3.356 = 4.305 \text{ kN}$$

The correction factor is

$$k = \frac{R}{F} = \frac{1.692}{4.305} = +0.393$$

The final end moments are obtained by multiplying the moments from the *sway* solution and then adding the moments from the *non-sway* solution to obtain the moments in the original structure.

Thus,

$$M_{AB} = 1.46 + 2.38(0.393) = +2.40 \text{ kN m}$$

$$M_{BA} = -2.12 + 2.36(0.393) = -1.19 \text{ kN m}$$

$$M_{CB} = -0.48 - 4.48(0.393) = -2.24 \text{ kN m}$$

$$M_{DC} = 0.24 + 5.59(0.393) = 2.44 \text{ kN m}$$

Frames with Inclined Members

The superposition method described for the case of rectangular frames with sidesway, can also be used in the analysis of frames with inclined legs. For example, for the single storey portal frame with inclined legs shown in Fig. 4.11, the frame is held against sidesway by artificial joint restraint R (Fig. 4.11(b)) and, after the moment distribution calculations are carried out, the value of R is determined. In the second analysis, the consistent joint force F is computed due to an arbitrary lateral displacement Δ.

After determining the consistent joint force F, the correction factor k is then obtained to calculate the final moments.

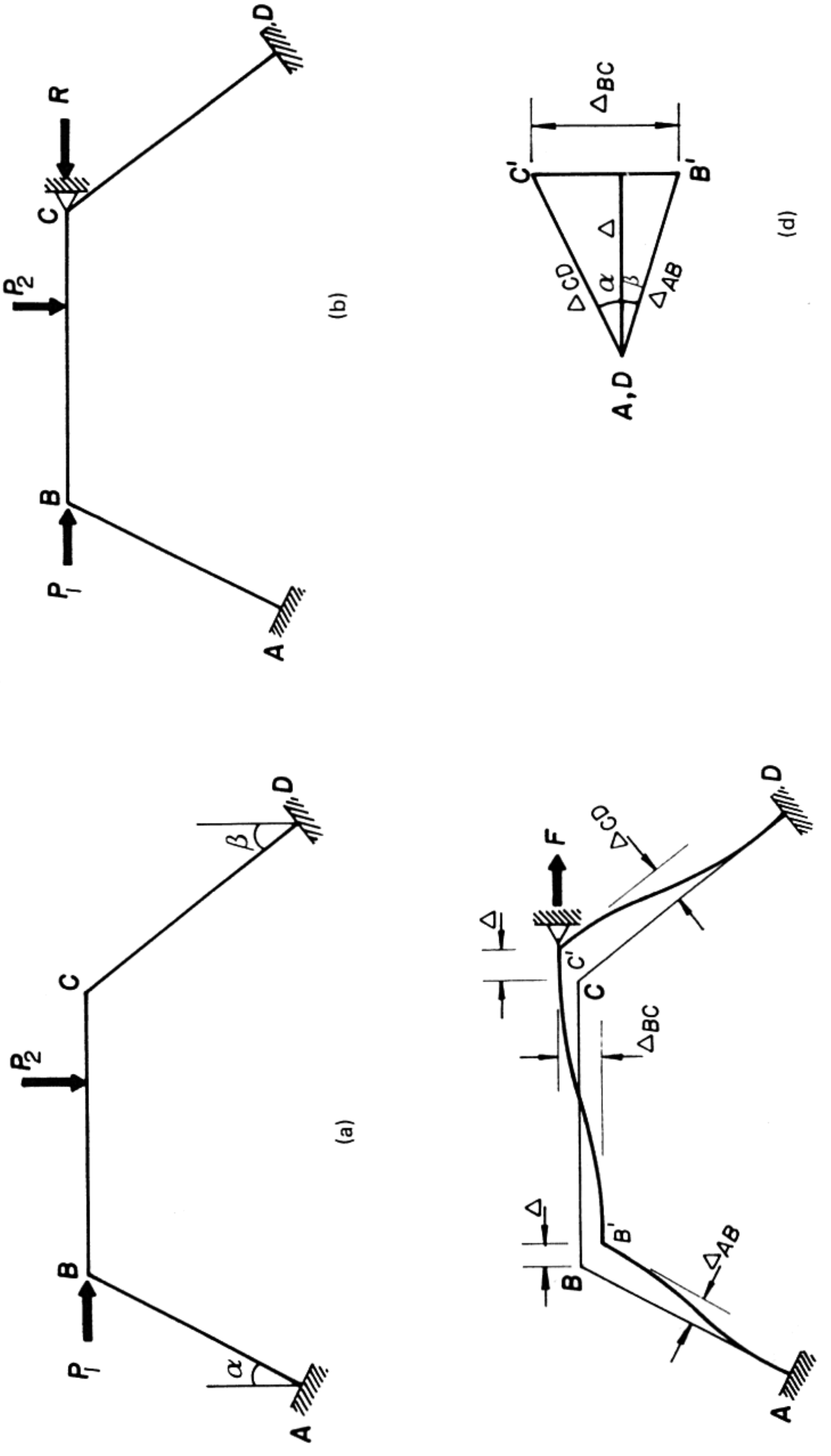


Figure 4.11

EXAMPLE 4.5 *Find the joint moments of the frame shown in Fig. 4.12 by the moment distribution method.*

Relative Stiffness and Distribution Factors

$$K_{AB} = \frac{30}{3} = 10$$

$$K_{BC} = \frac{30 \times 2}{4} = 15$$

$$K_{CD} = \frac{30}{5} = 6$$

$$(DF)_{BA} = \frac{10}{25} = 0.4$$

$$(DF)_{BC} = \frac{15}{25} = 0.6$$

$$(DF)_{CB} = \frac{15}{21} = 0.714$$

$$(DF)_{CD} = \frac{6}{21} = 0.286$$

Fixed-End Moments

$$M^F_{BC} = -M^F_{CB} = \frac{(10)(4)}{8} = 5.0 \text{ kN}$$

(a) Moment Distribution without Sidesway

The member shears are

$$V_{AB} = \frac{1.51 + 3.04}{3} = 1.52 \text{ kN}$$

$$V_{DC} = \frac{1.04 + 2.08}{5} = 0.62 \text{ kN}$$

Figure 4.12(b) shows the forces and reactions on the frame. The artificial

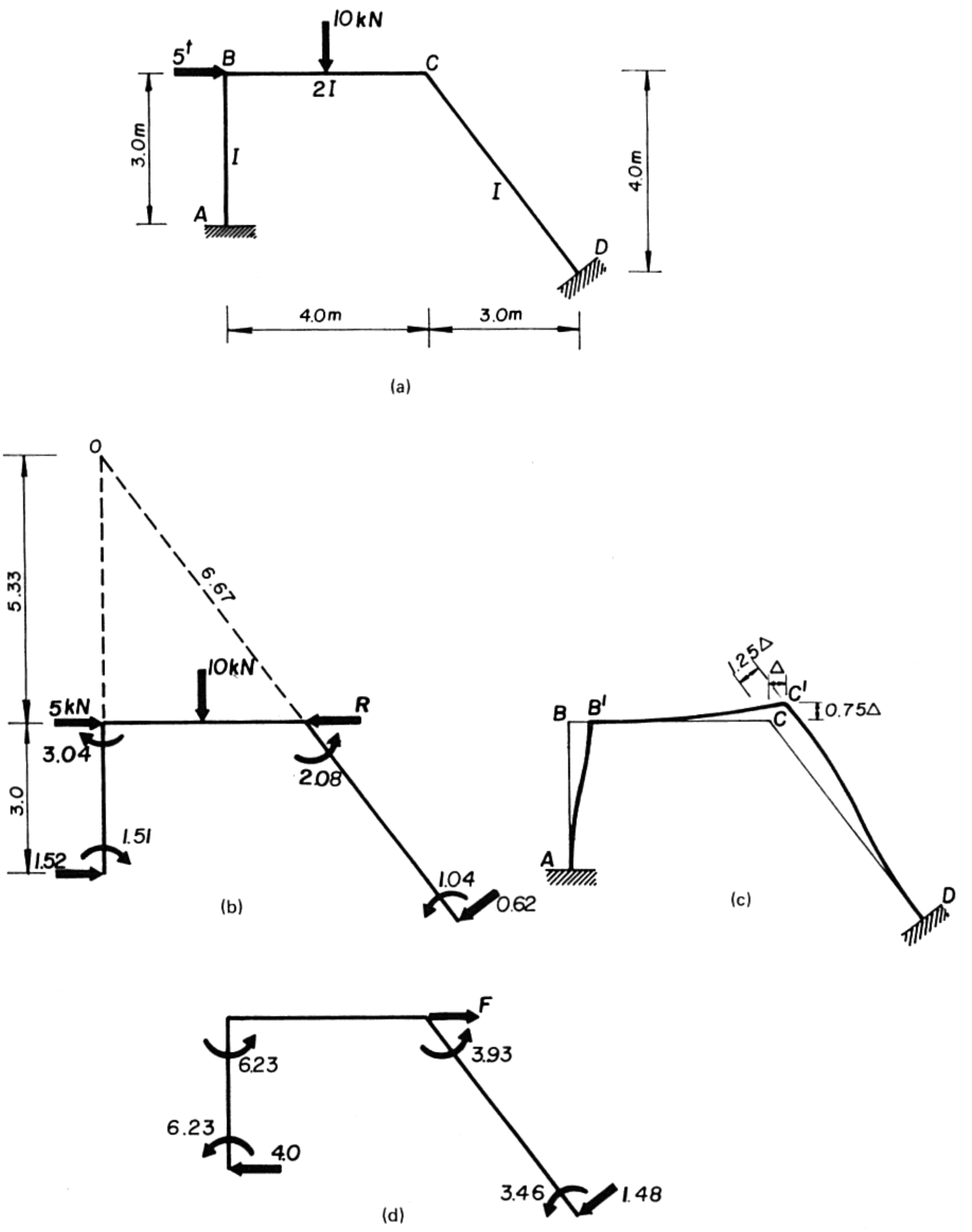


Figure 4.12

joint restraint R may be found by taking moments about the point of intersection of members AB and DC. Thus,

$$5.33R + 10(2) + 0.62(11.67) + 1.51 - 5(5.33) - 1.52(8.33) - 1.02 = 0$$

$$R = 2.17 \text{ kN}$$

Table 4.9 Distribution without Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	10	10	15	15	6	6
DF	0.0	0.40	0.60	0.714	0.286	0.0
Fixed-end moment			+5.0	−5.0		
	−1.0	−2.0	−3.0	−1.5		
			+2.32	+4.64	+1.86	+0.93
	−0.46	−0.93	−1.39	−0.70		
			+0.25	+0.50	+0.20	+0.10
	−0.05	−0.10	−0.15	−0.08		
			+0.03	+0.06	+0.02	+0.01
		−0.01	−0.02			
Total	−1.51	−3.04	+3.04	−2.08	+2.08	+1.04

(b) Moment Distribution with Sidesway

The loads are removed and the frame is permitted to sway to the right through a horizontal distance Δ. As shown in Fig. 4.12(c) the relative displacements are

Member AB : 1.0Δ

Member BC : 0.75Δ

Member CD : 1.25Δ

Fixed-End Moments

$$M_{AB}^F = \frac{6EI\Delta}{L^2} = \frac{6EI\Delta}{(3)^2}$$

$$M_{BC}^F = -\frac{6E(2I)(0.75\Delta)}{(4)^2}$$

$$M_{CD}^F = \frac{6EI(1.25\Delta)}{(5)^2}$$

Assuming $EI\Delta = 10$

$$M_{AB}^F = +6.67 \text{ kN m}$$

$$M_{BC}^F = -5.63 \text{ kN m}$$

$$M_{CD}^F = +3.00 \text{ kN m}$$

Table 4.10 Distribution with Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0.0	0.40	0.60	0.714	0.286	0.0
Fixed-end moment	+6.67	+6.67	−5.63	−5.63	+3.00	+3.00
	−0.21	−0.42	−0.62	−0.31		
			+1.05	+2.10	+0.84	+0.42
	−0.21	−0.42	−0.63	−0.31		
			+0.11	+0.22	+0.08	+0.04
	−0.02	−0.05	−0.07	−0.03		
				+0.02	+0.01	
Total	+6.23	+5.78	−5.79	−3.94	+3.93	+3.46

The member shears are

$$V_{AB} = \frac{6.23 + 5.78}{3} = 4.00 \text{ kN} \quad \text{(left)}$$

$$V_{DC} = \frac{3.46 + 3.93}{5} = 1.48 \text{ kN} \quad \text{(left)}$$

The consistent joint force F required to produce the set of moments given in Table 4.10 is found by taking moments about the point of intersection of AB and CD. Thus,

$$5.33F + 6.23 + 3.46 - 4(8.33) - 1.48(11.67) = 0$$

$$F = 7.67 \text{ kN}$$

The correction factor is

$$k = \frac{R}{F} = \frac{2.17}{7.67} = 0.283$$

Final Moments

The final moments are determined by algebraically adding the results of the no-sway solution to the products of k and the corresponding results of the sway solution. Thus

$$M_{AB} = -1.51 + 0.283(+6.23) = +0.25 \text{ kN m}$$

$$M_{BA} = -M_{BC} = -3.04 + 0.283(+5.78) = -1.40 \text{ kN m}$$

$$M_{CB} = -M_{CD} = -2.08 + 0.283(-3.94) = -3.20 \text{ kN m}$$

$$M_{DC} = +1.04 + 0.283(+3.46) = +2.02 \text{ kN m}$$

4.6.3 Frames With Multiple Degrees of Freedom With Respect to Sidesway

A rigid frame which has n independent joint translations is said to have *n degrees of freedom* with respect to sidesway. For example, in the case of the two-storey frame shown in Fig. 4.13(a), with joint B deflecting Δ_B and joint C an independent deflection of Δ_C, the frame is said to have two degrees of freedom with respect to sidesway.

By applying the principle of superposition, the two-storey frame (Fig. 4.13) may be analysed in three separate steps. First, the frame is completely prevented from sidesway by introducing artificial supports as shown in Fig. 4.13(b). In this case all the given loads are applied and a regular no-sway moment distribution is carried out to obtain the artificial joint restraint (assumed positive to the left) R_{10} and R_{20}. In the second step, a translation of the frame of an arbitrary displacement Δ_1 is introduced at joint C (Fig. 4.13(c)). While translation is introduced the joints are locked against rotation and initial moments are developed in members AB and EF as shown by the solid lines. To permit the joints to rotate, shown by the dotted elastic curve, a moment distribution is performed and the consistent joint forces R_{11} and R_{12} are calculated. Finally, a similar solution is conducted for an arbitrary displacement Δ_2 at joint C (Fig. 4.13(d)).

With the separate trial solutions, it now remains to determine how much of the two sidesway solutions should be superimposed to the first case to obtain the final results for the actual given problem. It must be possible to find the final values by obtaining multiplying factors k_1 and k_2 from a linear combination of the obtained results. These factors are obtained by solving two simultaneous equations formulated from the superposition equations for the reactions at D and E:

$$\begin{aligned} R_{10} + k_1 R_{11} + k_2 R_{12} &= 0 \\ R_{20} + k_1 R_{21} + k_2 R_{22} &= 0 \end{aligned} \qquad [4.12]$$

After finding the values of the proportionality factor k_1 and k_2, the moments in

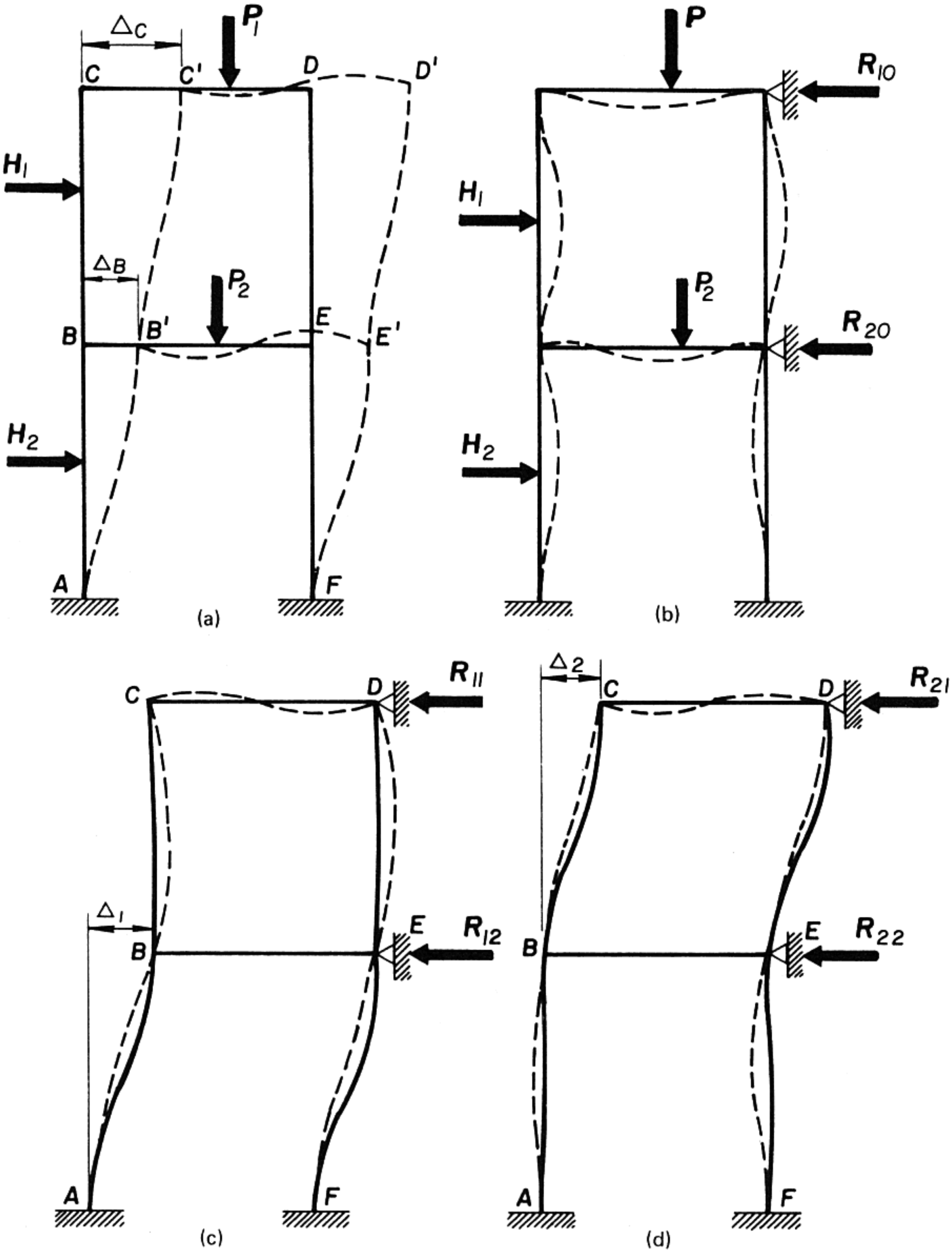


Figure 4.13

the frame are determined from the equation

$$M = M_0 + k_1M_1 + k_2M_2 \qquad [4.13]$$

where M_0 represents the moments from the non-sway moment distribution, and M_1 and M_2 are the moments due to sidesway, shown in Fig. 4.13(c) and (d) respectively.

The procedure described above can also be used in the analysis of a frame having ***n degrees of freedom*** with respect to sidesway. For such a frame there will be one non-sway moment distribution and ***n*** separate sidesway cases, in each case only *one* independent sidesway is permitted. After completing the required $(n + 1)$ moment distribution analyses, the ***n*** superposition equations for the artificial joint restraints and consistent joint are

$$\begin{bmatrix} R_{10} \\ R_{20} \\ \cdots \\ R_{30} \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ R_{n1} & R_{n2} & & R_{nn} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \cdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \qquad [4.14]$$

or the simultaneous equations may be written in index notation as

$$R_{i0} + \sum_{j=1}^{n} k_i R_{ij} = 0 \qquad i = 1, 2, \ldots n \qquad [4.15]$$

The solution of the above simultaneous equations gives the values of the multiplying factors $k_1, k_2, \ldots k_n$. These values are used to find the final moments.

$$M = M_0 + k_1 M_1 + k_2 M_2 + \ldots + k_n M_n$$

$$= M_0 + \sum_{j=1}^{n} k_j M_j \qquad [4.16]$$

Split-Level Frames

The frame of Fig. 4.14 has two degrees of freedom with respect to sidesway. The horizontal displacements of the two roof levels are designated as Δ_B and Δ_D as shown in Fig. 4.14(a). Since Δ_B and Δ_D are unknown, two separate solutions must be obtained where independent sidesways are permitted as shown in Fig. 4.14(c) and (d). With the consistent joint forces R_{10}, R_{12}, R_{21} and R_{22} determined, the proportionality factors k_1 and k_2 are obtained from equilibrium equations as described above.

Gabled Frames

Gabled frames of a single span have two degrees of freedom with respect to joint translation; accordingly, two artificial joint restraints are required to prevent the joints from moving. The steps required for the analysis of gabled frames by moment distribution are shown in Fig. 4.15.

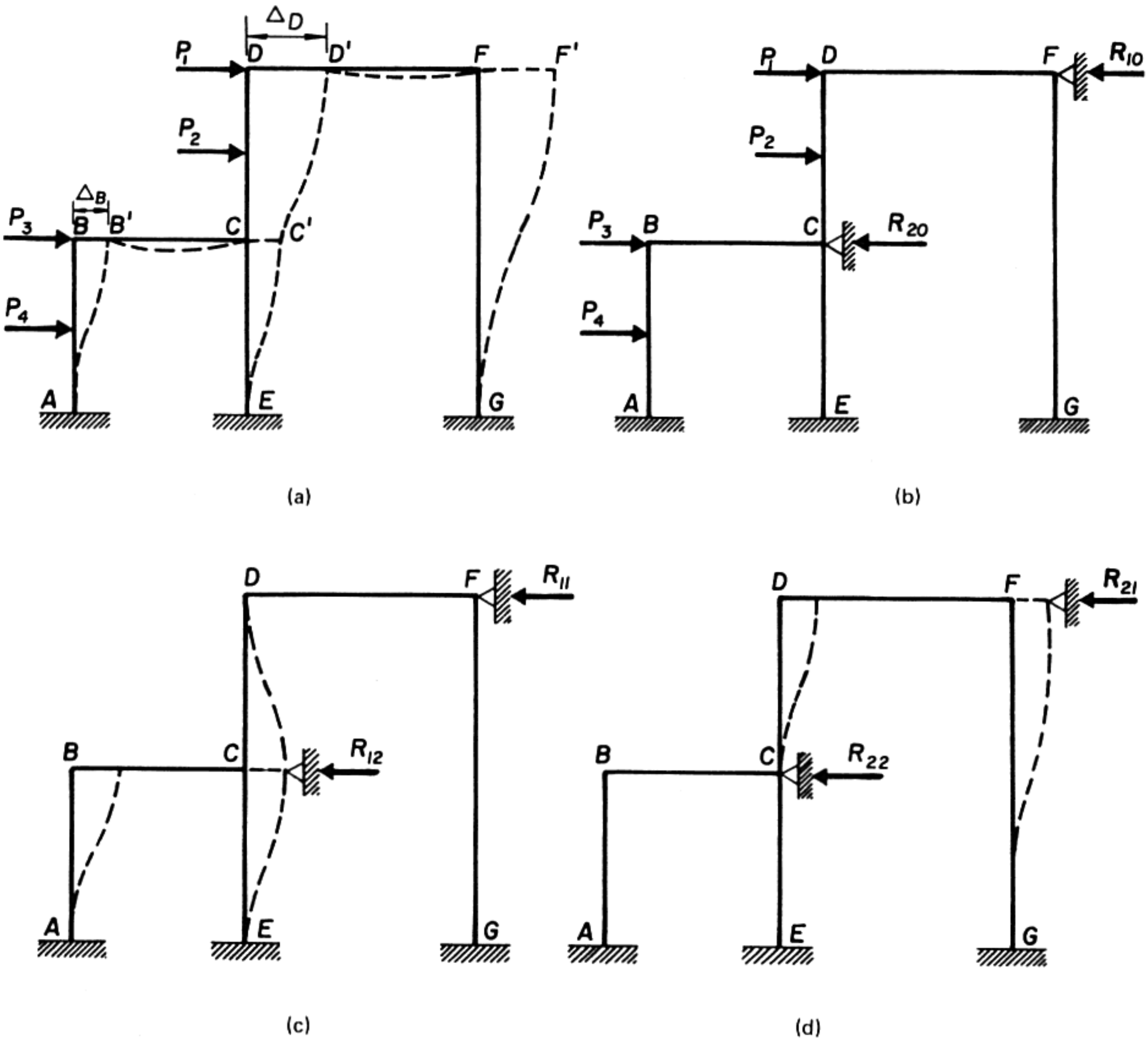


Figure 4.14

4.7 CANTILEVER MOMENT DISTRIBUTION

Unlike the conventional moment distribution, the cantilever moment distribution permits sidesway to occur during the moment balancing process. The method, therefore, makes it possible to evaluate the moments in one distribution without requiring artificial joint restraints. Within its range of applicability the cantilever moment distribution provides a simple but powerful method of analysing *symmetric frames* subjected to lateral loads at storey heights or to *antisymmetric loads*. The derivation of the basic equations necessary for the development of the cantilever moment distribution is given below.

(a) Stiffness Factor of a Cantilever

Consider a uniform cantilever beam AB subjected to an end moment M_{AB} as shown in Fig. 4.16.

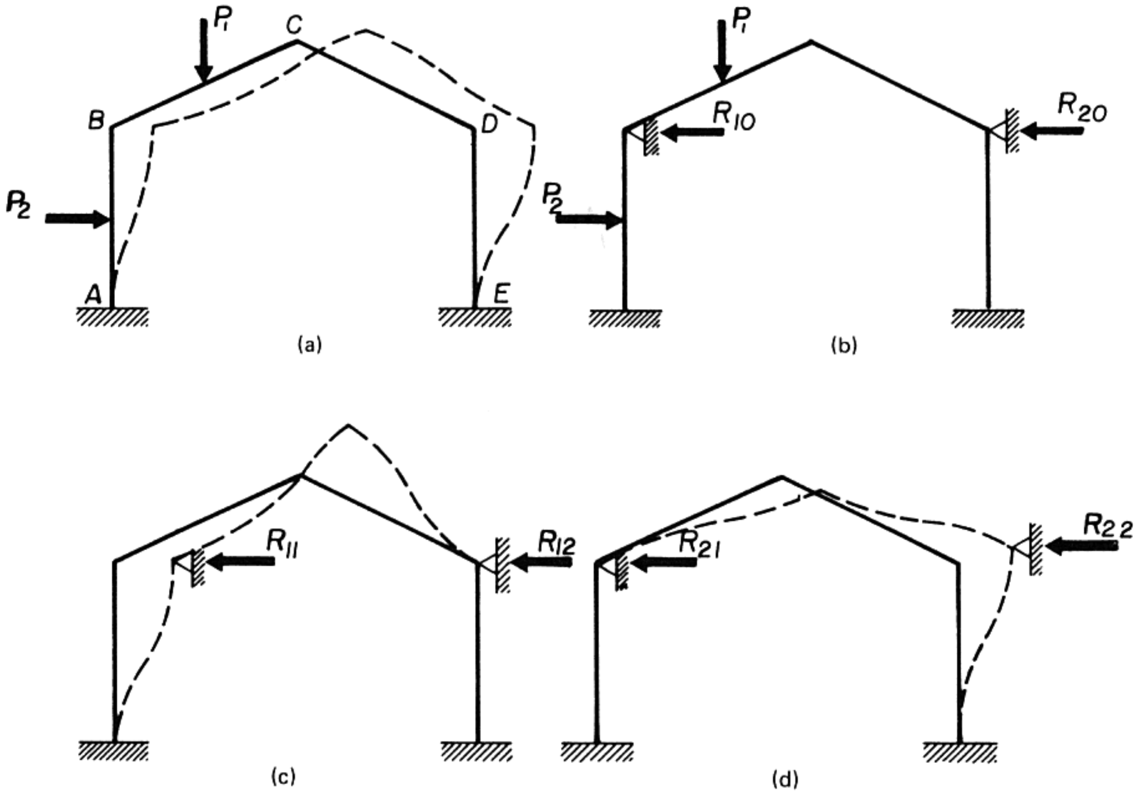


Figure 4.15

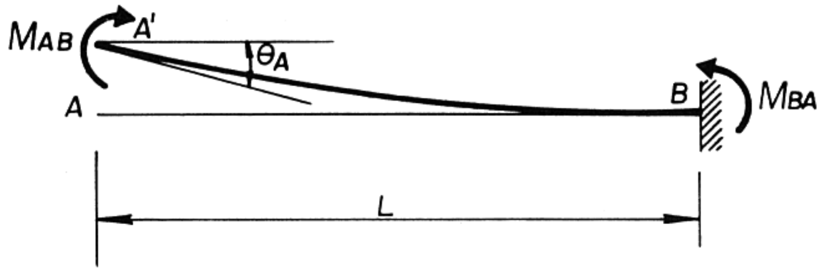


Figure 4.16

It is seen that the angle of rotation is given by the expression

$$\theta_{AB} = \frac{M_{AB}L}{EI}$$

or [4.17]

$$\frac{M_{AB}}{\theta_{AB}} = \frac{EI}{L}$$

which is defined as the *rotation stiffness* of the cantilever beam. Notice that the stiffness of beam AB is one-fourth of the same beam with support A hinged.

(b) Carry-over Factor

The bending moment at end B of the beam (Fig. 4.16), in accordance with the statical sign convention, is equal in magnitude but opposite in direction to the applied moment at support A. The translational carry-over factor is therefore −1.0.

(c) Stiffness Factor of Beams under Antisymmetric Bending

Consider a uniform beam (Fig. 4.17) under antisymmetric bending moments at the ends.

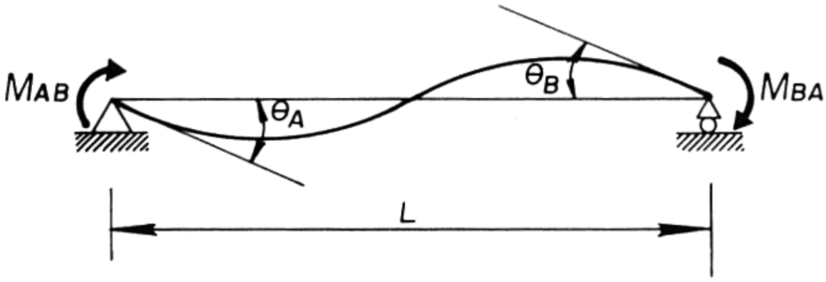


Figure 4.17

The slope-deflection equation for the beam is

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B)$$

or [4.18]

$$\frac{M_{AB}}{L} = \frac{6EI}{L}$$

which is the *rotation stiffness* of a simply supported beam under antisymmetric end moments.

The steps in applying the cantilever moment distribution method may be summarised as follows:

(a) Evaluate the stiffness value for a member parallel to the axis of symmetry from EI/L. The carry-over factor is −1.0 when the far end of the member is fixed.
(b) The stiffness factor for a member perpendicular to the axis of symmetry is $6EI/L$.
(c) Compute the fixed-end moments from the condition that the joints are locked against rotation but free to translate. For more than one-storey

frames, the fixed-end moments are computed to be directly proportional to the product of the storey-shear and storey-height.

(d) Balance the moments for one-half of the structure.
(e) Determine the correction factor to satisfy horizontal equilibrium condition.
(f) Compute final joint moments by multiplying with the correction factor the moments obtained in step (d).

EXAMPLE 4.6 *Determine the joint moments of the frame in Fig. 4.18 using (a) the conventional moment distribution method (b) the cantilever moment distribution method.*

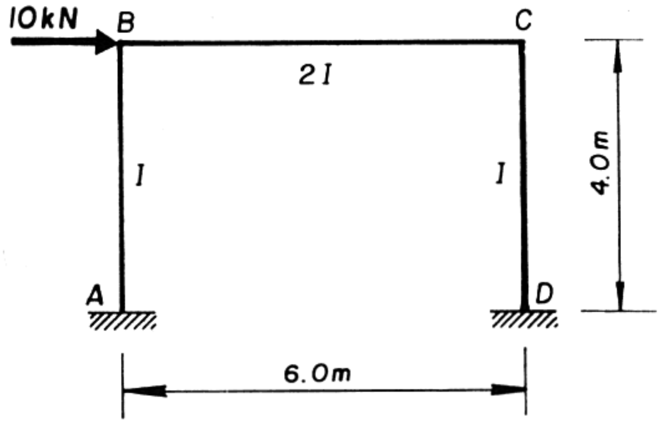


Figure 4.18

(a) Conventional Moment Distribution Method

Relative Stiffness Values and Distribution Factors

$$K_{AB} = K_{CD} = \frac{1}{4} = 0.25$$

$$K_{BC} = \frac{2}{6} = 0.33$$

$$(DF)_{BA} = (DF)_{CD} = \frac{0.25}{0.25 + 0.33} = 0.431$$

$$(DF)_{BC} = (DF)_{CB} = \frac{0.33}{0.58} = 0.569$$

Fixed-End Moments (Relative Values)

$$M^F_{AB} = M^F_{BA} = M^F_{CD} = M^F_{DC} = \frac{6EI\Delta}{L^2} = 10.0 \text{ kN m}$$

Table 4.11

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	0.25	0.25	0.33	0.25	0.25	0.25
DF	0.0	0.431	0.569	0.569	0.431	
Fixed-end moment	+10.0	+10.0			+10.0	+10.0
			−2.85	−5.69	−4.31	−2.15
	−1.54	−3.08	−4.07	−2.04		
			+0.58	+1.16	+0.88	+0.44
	−0.13	−0.25	−0.33	−0.16		
			+0.04	+0.09	+0.07	+0.03
	−0.01	−0.02	−0.02			
Total	+8.32	+6.65	−6.65	−6.64	+6.64	8.32

The base shear is

$$V = \frac{1}{4.0}(M_{AB} + M_{BA} + M_{CD} + M_{DC})$$

$$= \frac{2}{4.0}(8.32 + 6.65) = 7.485 \text{ kN}$$

The correction factor is

$$k = \frac{10.0}{7.485} = 1.336$$

The final end moments are

$$M_{AB} = M_{DC} = 1.336 \times 8.32 = 11.12 \text{ kN m}$$

$$M_{BA} = M_{CD} = 1.336 \times 6.65 = 8.88 \text{ kN m}$$

(b) Cantilever Moment Distribution Method

Relative Stiffness Values and Distribution Factors

$$K_{AB} = K_{CD} = \frac{EI}{L} = \frac{1}{4.0} = 0.25$$

$$K_{\mathrm{CB}} = \frac{6EI}{L} = \frac{6 \times 2}{6.0} = 2.0$$

$$(DF)_{\mathrm{BA}} = \frac{2.0}{2.0 + 0.25} = 0.11$$

$$(DF)_{\mathrm{BC}} = \frac{2}{2.25} = 0.89$$

Fixed-End Moments (Relative Values)

$$M^F_{\mathrm{AB}} = M^F_{\mathrm{BA}} = \frac{6EI}{L^2} = 10.0 \text{ kN m}$$

The distribution is carried out in tabular form as shown in Table 4.12. Notice that the fixity at joint B does not exist, it is therefore released by applying at joint B a balancing moment of –10.0 kN m. The moment at joint B is distributed as –10.0 x 0.89 = –8.9 kN m to member BC and –10.0 x 0.11 = –1.1 kN m to member BA. The carry-over factor for column BA being –1.0, the moment carried over to joint A from B is +1.1 kN m.

Table 4.12

Joint	A	B	
Member	AB	BA	BC
DF	0.0	0.11	0.89
Fixed-end moment	+10.0 +1.1	+10.0 –1.1	–8.9
Total	–11.1	+8.9	–8.9

The base shear is

$$V = \frac{2}{4}(11.1 + 8.9) = 10.0 \text{ kN m}$$

The correction factor is

$$k = \frac{10.0}{10.0} = 1.0$$

The final joint moments are

$$M_{\mathrm{AB}} = M_{\mathrm{DC}} = 1.0(11.1) = 11.1 \text{ kN m}$$

$$M_{\mathrm{BA}} = M_{\mathrm{CD}} = 1.0(8.9) = 8.9 \text{ kN m}$$

It is noted that the computations are carried out on one-half of the frame only since the moments on either sides of the axis of symmetry of the frame are identical.

EXAMPLE 4.7 *Find the joint moments of the two-storey frame shown in Fig. 4.19 using the cantilever moment distribution method.*

Relative Stiffness Values and Distribution Factors

$$K_{AB} = K_{EF} = \frac{2I}{4} = 0.50I$$

$$K_{BC} = K_{DE} = \frac{I}{3} = 0.33I$$

$$K_{BE} = K_{CD} = \frac{6(2I)}{4} = 3.0I$$

$$(DF)_{BA} = \frac{0.50}{0.50 + 0.33 + 3.0} = 0.130$$

$$(DF)_{BE} = \frac{3.0}{3.83} = 0.783$$

$$(DF)_{BC} = \frac{0.33}{0.83} = 0.087$$

$$(DF)_{CB} = \frac{0.33}{0.33 + 3.0} = 0.10$$

$$(DF)_{CD} = \frac{3}{3.33} = 0.90$$

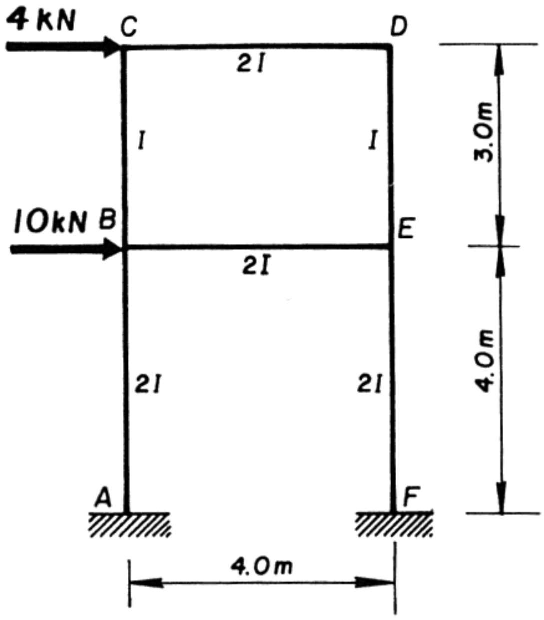


Figure 4.19

Fixed-End Moments (Relative Values)

The fixed-end moments are taken as the product of the storey-shear and the storey-height since the columns behave as a cantilever. Therefore,

$$M_{BC}^{F} = M^{F}$$

Subsequently, in the upper storey the fixed-end moments are found by multiplying the storey-shear (4.0 kN) by the column height (3.0 m). In the lower storey the total horizontal force is 14.0 kN.

Thus

$$M_{BC}^{F} = M_{CB}^{F} = M_{ED}^{F} = M_{DE}^{F} = 4 \times 3 = 12 \text{ kN m}$$

$$M_{AB}^{F} = M_{BA}^{F} = M_{EF}^{F} = M_{FE}^{F} = 14 \times 4 = 56 \text{ kN m}$$

Table 4.13

Joint	A	B			C	
Member	AB	BA	BE	BC	CB	CD
K	0.50	0.50	3.0	0.33	0.33	3.0
DF	0.0	0.130	0.783	0.087	0.10	0.90
Fixed-end moment	+56.00	+56.00		+12.00	+12.00	
	+8.84	−8.84	−53.24	−5.92	+5.92	
				+1.79	−1.79	−16.13
	+0.23	−0.23	−1.40	−0.16	+0.16	
				+0.02	−0.02	
		−0.00	−0.02	−0.00		
Total	+65.07	+46.93	−54.66	+7.73	+16.27	−16.27

Correction Factor

Upper storey shear

$$V_1 = \frac{1}{3}(M_{CB} + M_{BC} + M_{DE} + M_{ED})$$

$$= \frac{2}{3}(7.73 + 16.27) = 16 \text{ kN}$$

Lower storey shear

$$F_2 = \frac{2}{4}(65.07 + 46.93) = 56 \text{ kN}$$

A correction factor must be applied at each level to satisfy horizontal equilibrium.

Since the relative fixed-end moments are taken as the product of storey-shear and storey-height, the correction factor becomes common to both storeys and is given as

$$\text{correction factor} = \frac{4}{16} = \frac{14}{56} = 0.25$$

The final end moments are obtained by multiplying those obtained in Table 4.13 with the correction factor.

Final End Moments

$$M_{AB} = 0.25(+65.07) = +16.27 \text{ kN m}$$

$$M_{BA} = 0.25(+46.93) = +11.73 \text{ kN m}$$

$$M_{BE} = 0.25(-54.66) = -13.67 \text{ kN m}$$

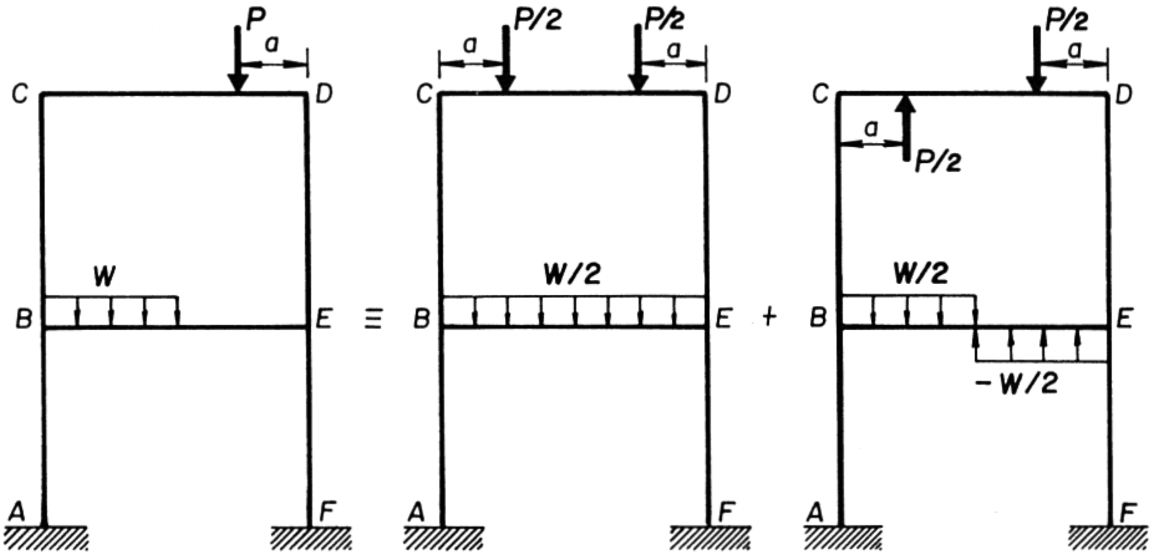
$$M_{BC} = 0.25(+7.73) = +1.93 \text{ kN m}$$

$$M_{CB} = 0.25(+16.27) = +4.07 \text{ kN m}$$

$$M_{CD} = 0.25(-16.27) = -4.07 \text{ kN m}$$

4.8 ARBITRARY LOADING ON SYMMETRIC FRAMES

Any arbitrary loading on a symmetric frame can be *resolved* into *symmetric* and *antisymmetric* loading system components. The frame may then be analysed by using the conventional moment distribution method to the symmetric loading system, and the cantilever moment distribution method for the frame subjected to the antisymmetric loading. The final end moments are then obtained by adding algebraically the results of the two solutions. Figure 4.20 shows a symmetric frame subjected to any arbitrary loading. The same frame is also shown loaded by symmetric and antisymmetric (Fig. 4.20(b) and (c)) loading systems whose algebraic sum furnishes an equivalent system to the original loading.



(a) Arbitrary loading (b) Symmetric loading (c) Antisymmetric loading

Figure 4.20

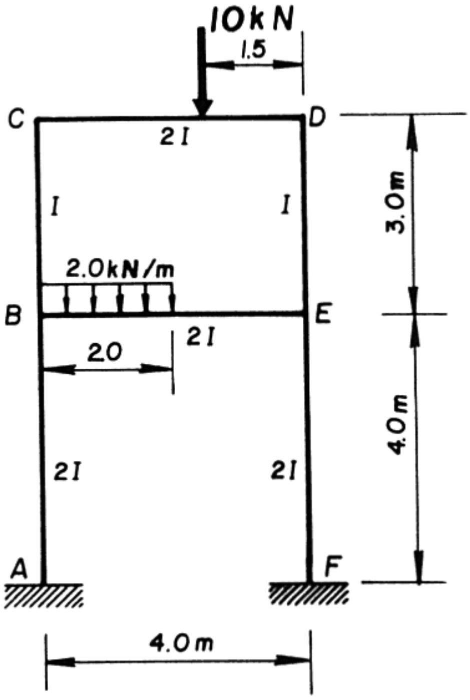


Figure 4.21

EXAMPLE 4.8 *Determine the joint moments of the two-storey frame in Fig. 4.21.*

(a) Symmetric Loading
The frame is analysed with the conventional moment distribution method on half of the frame only, since the frame and the loading are both symmetrical.

Relative Stiffness and Distribution Factor

$$K_{AB} = K_{EF} = \frac{2I}{4} = 0.50I$$

$$K_{BC} = K_{DE} = \frac{I}{3} = 0.33I$$

$$K_{BE} = K_{CD} = \frac{2I}{4} = 0.50I$$

$$(DF)_{BA} = (DF)_{BE} = \frac{0.50}{0.50 + 0.33 + 0.50} = 0.375$$

$$(DF)_{BC} = \frac{0.333}{1.333} = 0.25$$

$$(DF)_{CB} = \frac{0.333}{0.333 + 0.50} = 0.40$$

$$(DF)_{CD} = \frac{0.50}{0.833} = 0.60$$

Fixed-End Moments

$$M^F_{CD} = -M^F_{DC} = \frac{5(2.5)(1.5)^2}{(4.0)^2} + \frac{5(1.5)(2.5)^2}{(4.0)^2} = 4.69 \text{ kN m}$$

$$M^F_{BE} = -M^F_{ED} = \frac{1.0(4.0)^2}{12} = 1.33 \text{ kN m}$$

Table 4.14 shows the distribution of the fixed-end moments carried over the left half of the frame. The moments carried from the right half are indicated in *italics*.

Table 4.14

Joint	A	B			C	
Member	AB	BA	BE	BC	CB	CD
K	0.50	0.50	0.50	0.333	0.333	0.50
DF	0.0	0.375	0.375	0.250	0.40	0.60
Fixed-end moment			+1.33			+4.69
	−0.25	−0.50	−0.50	−0.33	−0.17	
			+0.25	−0.90	−1.81	−2.71
	+0.12	+0.24	+0.24	+0.17	+0.08	*+1.35*
			−0.12	−0.29	−0.57	−0.86
	+0.07	+0.14	+0.14	+0.09	+0.04	*+0.43*
			−0.03	−0.09	−0.19	−0.28
	+0.03	+0.06	+0.06	+0.05	+0.02	*+0.14*
			−0.03	−0.03	−0.06	−0.08
	+0.01	+0.02	+0.02	+0.02		
Total	−0.02	−0.04	+1.32	−1.27	−2.66	−2.66

(b) Antisymmetric Loading

The cantilever moment distribution method is used to analyse the frame.

Relative Stiffnesses and Distribution Factors

$$K_{AB} = \frac{2I}{4} = 0.50I$$

$$K_{BC} = \frac{I}{3} = 0.33I$$

$$K_{BE} = K_{CD} = \frac{6(2I)}{4} = 3.0I$$

$$(DF)_{BA} = \frac{0.50}{0.50 + 0.33 + 3.0} = 0.130$$

$$(DF)_{BC} = \frac{0.33}{0.83} = 0.087$$

$$(DF)_{BE} = \frac{3.0}{3.83} = 0.783$$

$$(DF)_{CB} = \frac{0.33}{0.33 + 3.0} = 0.10$$

$$(DF)_{BC} = \frac{3.0}{3.33} = 0.90$$

Fixed-End Moments

The fixed-end moment for a beam with a uniformly distributed load extended kL from the left end (support A) is

$$M_{AB}^F = wL^2 \left(\frac{k^2}{12}\right)(6 - 8k + 3k^2)$$

$$M_{BA}^F = -wL^2 \left(\frac{k^3}{12}\right)(4 - 3k)$$

For the problem at hand, $k = 0.5$. Thus

$$M_{BE}^F = 1.0(4.0)^2 \frac{(0.5)^2}{12} [6 - 8(0.5) + 3(0.5)^2]$$

$$- 1.0(4.0)^2 \frac{(0.5)^3}{12} (4 - 3(0.5))$$

$$= +0.500 \text{ kN m}$$

$$M_{CD}^F = -\frac{5(1.5)(2.5)^2}{(4.0)^2} + \frac{5(2.5)(1.5)^2}{(4.0)^2}$$

$$= -1.172 \text{ kN m}$$

These are distributed using the cantilever moment distribution method as shown in Table 4.15.

Final End Moments

$M_{AB} = -0.02 + 0.050 = +0.030$ kN m $M_{DC} = +2.66 - 0.156 = -2.504$ kN m

$M_{BA} = -0.04 - 0.50 = -0.090$ kN m $M_{DE} = +2.66 + 0.146 = +2.806$ kN m

$M_{BC} = -1.27 - 0.146 = -1.416$ kN m $M_{EB} = -1.32 + 0.196 = -1.124$ kN m

$M_{BE} = +1.32 + 0.196 = +1.516$ kN m $M_{ED} = +1.27 - 0.146 = +1.124$ kN m

$M_{CB} = -2.66 + 0.146 = -2.514$ kN m $M_{EF} = +0.04 - 0.050 = -0.10$ kN m

$M_{CD} = -2.66 - 0.156 = -2.816$ kN m $M_{FE} = +0.02 + 0.050 = +0.070$ kN m

Table 4.15

Joint	A	B			C	
Member	AB	BA	BE	BC	CB	CD
K	0.50	0.50	3.0	0.33	0.33	3.0
DF	0.0	0.130	0.783	0.087	0.10	0.90
			+0.500			−1.172
	+0.065	−0.065	−0.392	−0.043	+0.043	
				−0.113	+0.113	+1.016
	−0.015	+0.015	+0.088	+0.010	−0.10	
Total	+0.050	−0.050	+0.196	−0.146	+0.146	−0.156

4.9 PROBLEMS

4.1 Find the support moments of the continuous beam shown in Fig. P4.1.

(*Ans*: $M_A = -12.6$ kN m

$M_B = -14.4$ kN m

$M_C = -2.2$ kN m)

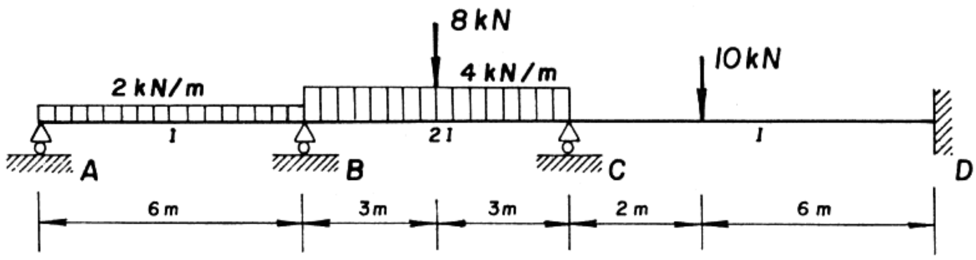


Figure P4.1

4.2 Determine the joint moments of the frame shown in Fig. P3.2.

4.3 Draw the bending moment of the frame shown in Fig. P4.3.

(*Ans*: $M_A = -14.72$ kN m

$M_B = +10.19$ kN m

$M_C = -6.73$ kN m)

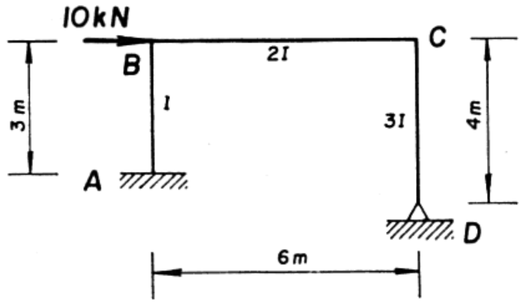


Figure P4.3

4.4 Calculate the support moments of the frame of Fig. P4.4.

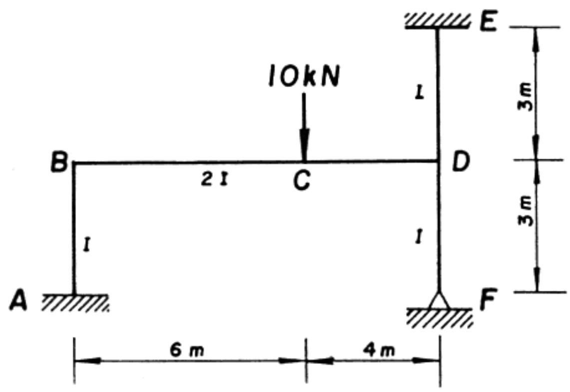


Figure P4.4

4.5 Find the joint moments of the split-frame shown in Fig. P4.5.

(*Ans*: M_A = –7.93 kN m

M_B = +6.08 kN m

M_C = –6.65; –1.69; –4.96 kN m

M_D = +7.37 kN m

M_E = +3.49 kN m

M_F = –2.48 kN m)

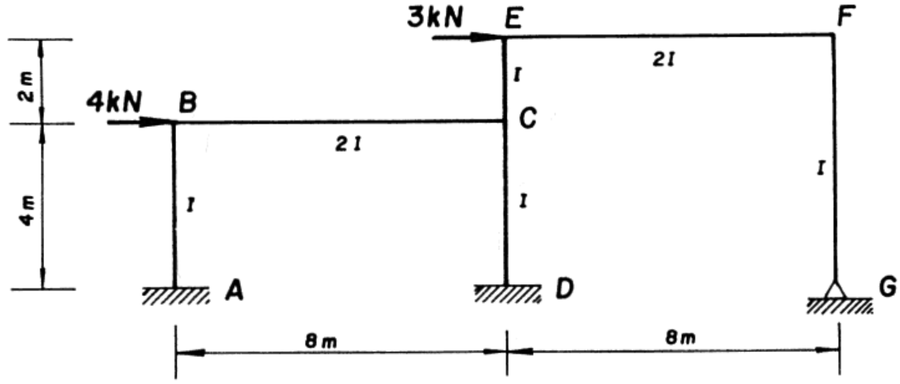


Figure P4.5

4.6 Find the joint moments of the two-storey frame shown in Fig. P4.6 using the cantilever moment distribution method.

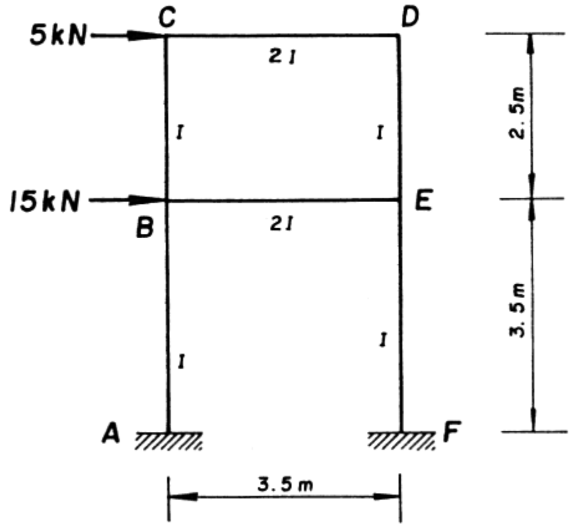


Figure P4.6

4.7 Find the joint moments of the two-storey frame shown in Fig. P4.7 using the cantilever moment distribution method.

(*Ans*: M_{AB} = +0.63 kN m　　M_D = –5.35 kN m

M_{BA} = +0.19 kN m　　M_{ED} = +1.69 kN m

M_{BE} = –3.25 kN m　　M_{EB} = +2.46 kN m

M_{BC} = +3.44 kN m　　M_{EF} = –0.78 kN m

M_C = –3.6 kN m　　M_{FE} = –0.34 kN m)

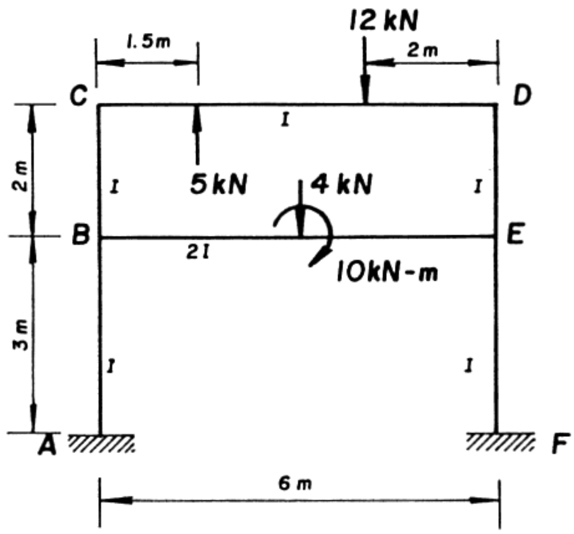


Figure P4.7

5. Kani Method of Moment Distribution

5.1 INTRODUCTION

The moment distribution by the Cross and Kani methods are both iterative procedures to solve the slope deflection equations. However, the Cross method obtains the unknown end moments by iterating the moment increments, while Kani's method iterates the end moments themselves as the unknowns. Kani's method consists of carrying out a single operation applied repeatedly at the joints of a structure in an arbitrary sequence. Using Kani's method the results may also be obtained to any desired accuracy by continuing the calculations a sufficient number of times. In addition to its simplicity, the method has the advantage of acquiring a *built-in* error elimination scheme. Moreover, the method is more suitable for frames with high degree of redundancy, including frames with sidesway, since only one set of computations is necessary. For such frames the required computational effort by Kani method is minimal when compared to other methods.

5.2 FRAMES WITHOUT SIDESWAY

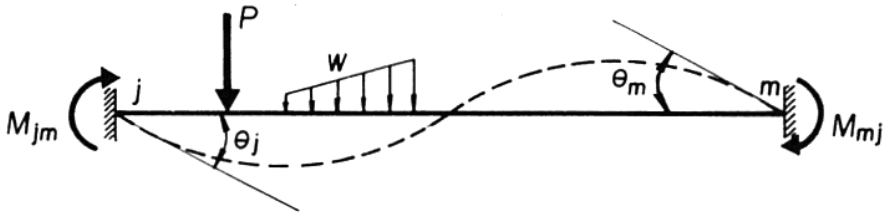
Consider member j–m as integral part of a frame (Fig. 5.1) and that there are many such members meeting at joint j so that m is the general designation, for the far ends. Let M_{jm} and M_{mj} be the end moments due to the applied loads at joints j and m, respectively.

The slope deflection equation for member j–m at joint j which is considered part of a frame without sidesway is written as

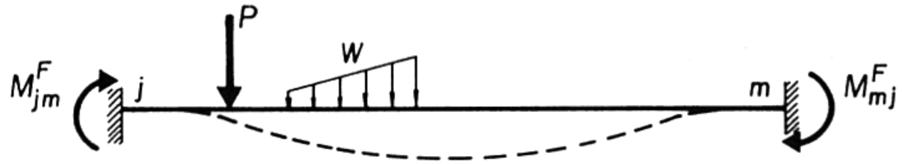
$$M_{jm} = M_{jm}^F - 2EK_{jm}(2\theta_j + \theta_m) \qquad [5.1]$$

The same equation may be written in the form

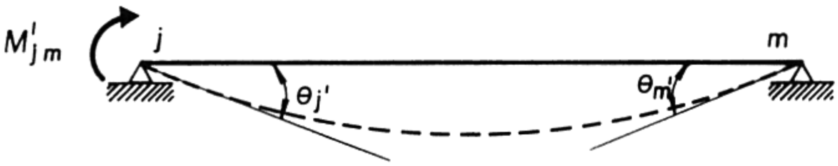
$$M_{jm} = M_{jm}^F + 2M'_{jm} + M'_{mj} \qquad [5.2]$$



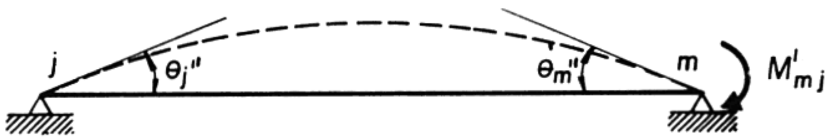
(a) Loading system



(b) Fixed-end moments



(c) Moment at *j*



(d) Moment at *m*

Figure 5.1

where

$$M'_{jm} = -2EK_{jm}\theta_j \qquad\qquad M^F_{jm}\ {}^{\top}\Sigma m$$

$$M'_{mj} = -2EK_{jm}\theta_m$$

Since M'_{jm} constitutes the contribution by θ_j to the total moment M_{jm} it is referred to as the *rotation contribution* of the end m to M_{jm}.

For any joint j where the number of m members are joined to it, the joint being in equilibrium of end-moments gives

$$\sum_m M_{jm} = 0$$

or

$$\sum_m M^F_{jm} + \sum_m (2M'_{jm} + M'_{mj}) = 0 \qquad [5.3]$$

Defining M_j the algebraic sum of the fixed-end moments at joint j as the *restraint moment*

$$M_j = \sum_m M^F_{jm}$$

Then [5.3] may be written as

$$\sum_m M'_{jm} = -\frac{1}{2}(M_j + \sum_m M'_{mj}) \qquad [5.4]$$

Since M_{jm} for any member must be proportional to the relative stiffness of the member, the moment in any member j–m is

$$M'_{jm} = \frac{K_{jm}}{\Sigma K_{jm}} \sum_m M'_{jm}$$

$$= -\frac{1}{2}\left(\frac{K_{jm}}{\Sigma K_{jm}}\right)(M_j + \sum_m M'_{mj}) \qquad [5.5]$$

Denoting the term $-\frac{1}{2}(K_{jm}/\Sigma K_{jm})$ as the *rotation factor* R_{jm}, [5.5] may be written as

$$M'_{jm} = R_{jm}(M_j + \sum_m M'_{mj}) \qquad [5.6]$$

Equation [5.6] forms the basis for Kani's method for frames without sidesway where the rotation contributions are evaluated for member j–m. These contributions and the fixed-end moments are then added algebraically to determine M_{jm} (see [5.1]). But M'_{jm} must initially be evaluated. However, their final values may be determined by successive approximations using the Gauss–Seidel iteration scheme where the rotation contributions proceed from estimated values (starting with initial zero-approximation), the subsequent values being obtained with better approximation. The iteration is terminated when the latest approximation furnishes a value acceptably close to the preceding result.

Kani's method for frames without sidesway may be summarised as follows:

(a) Determine.the fixed-end moments of all members. At each joint evaluate the resultant fixed-end moment, *restraint moment*, ΣM_j.
(b) Calculate the *relative stiffnesses* (K values) of all members using the Gauss–Seidel iteration scheme.
(d) Evaluate the final end-moments using [5.2].

Hinged-End Members

In a structure which contains a hinge located at one end, the stiffness of the member becomes *three-fourths* of the stiffness of a corresponding beam fixed at both ends. Such members, after the fixed-end moments are determined based on the actual member length and introduced into the calculation scheme, are replaced by fictitious members fixed at the hinged ends with the K values which are three-fourths of the actual K values of the members in the original structures. This substitution is justified since the end moment required to produce a unit rotation in the original member is the same as the substitute with three-fourths stiffness. The hinged-end is then left free and its final end-moment is set to zero.

EXAMPLE 5.1 *Find the joint end moments of the symmetric frame shown in Fig. 5.2.*

Fixed-End Moments

$$M^F_{DE} = M^F_{EF} = -M^F_{ED} = -M^F_{FE} = \frac{3(6)^2}{12} = 9.0 \text{ kN m}$$

$$M^F_{GH} = M^F_{HI} = -M^F_{HG} = -M^F_{IH} = \frac{2(6)^2}{12} = 6.0 \text{ kN m}$$

The fixed-end moments are recorded at the corresponding member ends in the scheme of calculation (Fig. 5.3).

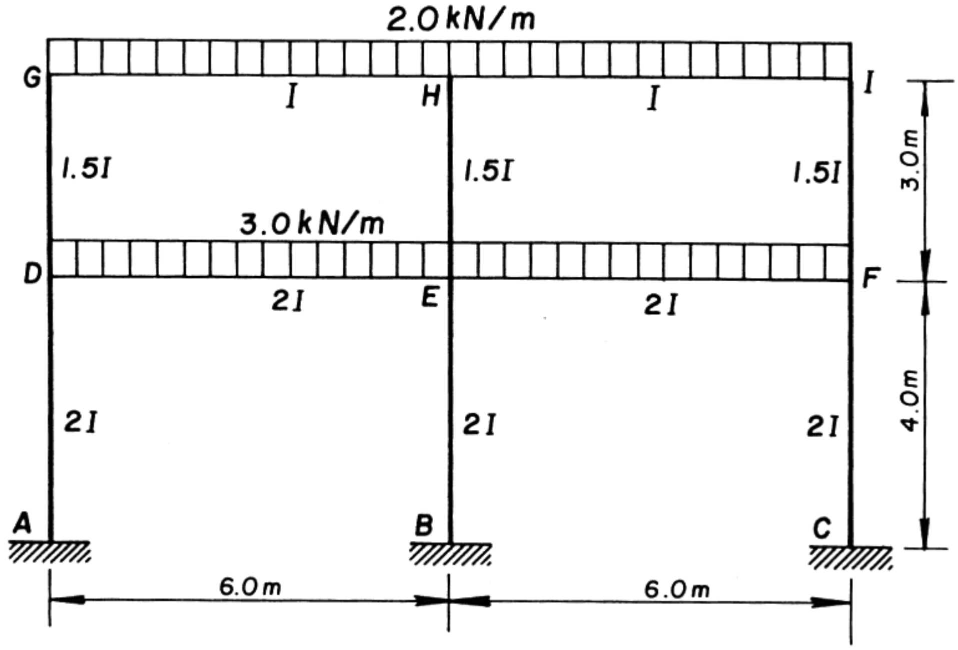


Figure 5.2

Restraint Moments ($M_j = \sum_m M^F_{jm}$)

$M_G = +6.0$ kN m

$M_H = (-6.0 + 6.0) = 0.0$

$M_I = -6.0$ kN m

$M_D = +9.0$ kN m

$M_E = 0.0$

$M_F = -9.0$ kN m

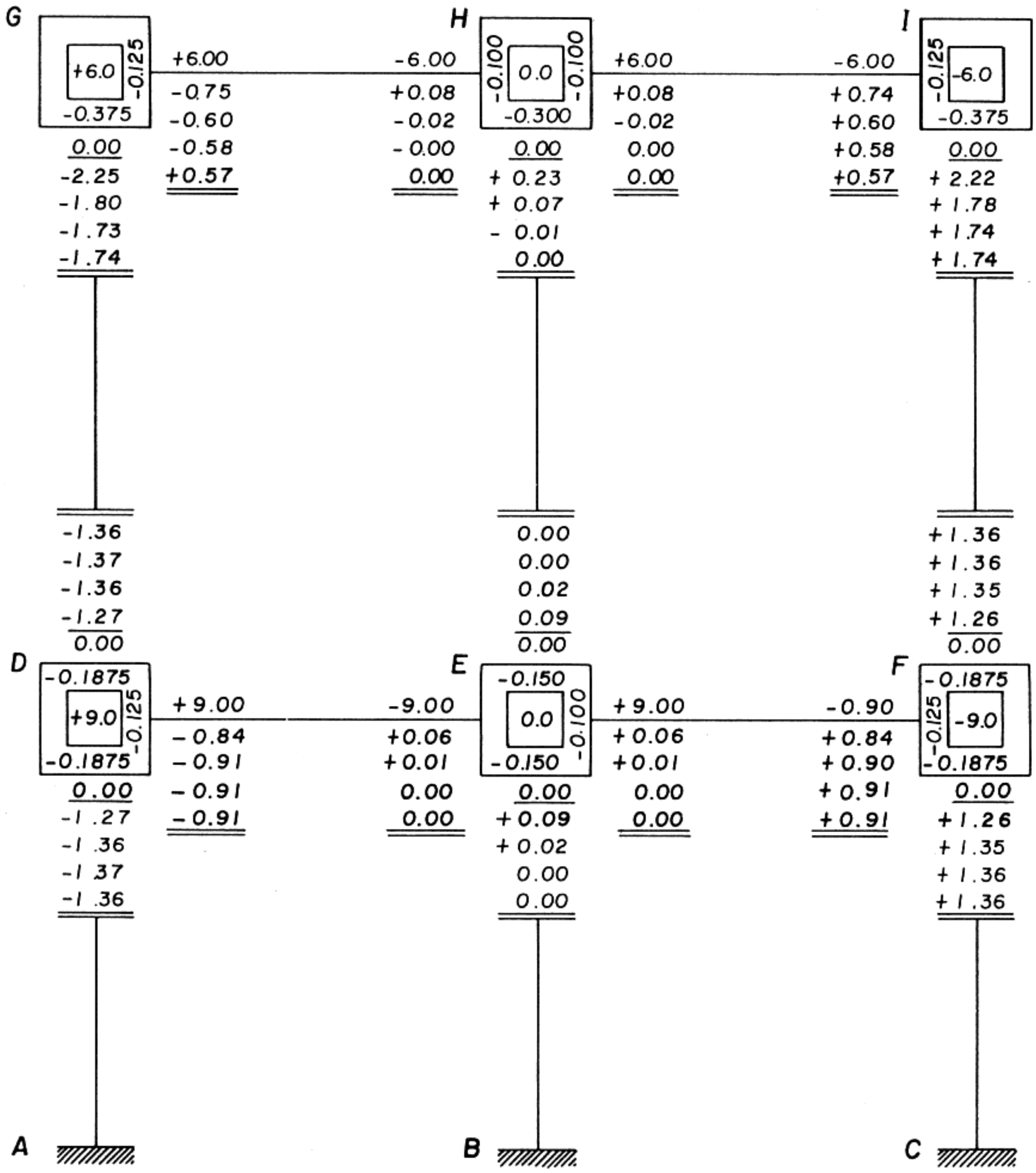


Figure 5.3

Relative Stiffness Values $(K_{jm} = I_{jm}/L_{jm})$

$$K_{AD} = K_{BE} = K_{CF} = \frac{2}{4} = 0.50$$

$$K_{DG} = K_{EH} = K_{FI} = \frac{1.5}{3} = 0.50$$

$$K_{DE} = K_{EF} = \frac{2}{6} = 0.333$$

$$K_{GH} = K_{HI} = \frac{1}{6} = 0.167$$

Rotation Factors ($R_{jm} = -0.5K_{jm}/\sum_m K_{jm}$)

$$R_{DA} = -\frac{0.50(0.5)}{0.5 + 0.333 + 0.50} = -0.1875$$

$$R_{DG} = -\frac{0.5(0.5)}{1.333} = -0.1875$$

$$R_{DE} = -\frac{0.5(0.333)}{1.333} = -0.125$$

Check. $\Sigma R_{jm} = -0.1875 - 0.1875 - 0.125 = -0.50$

Similarly

$R_{GD} = -0.375$ $\quad R_{GH} = -0.125$

$R_{HG} = -0.10$ $\quad R_{HE} = -0.30$ $\quad R_{HI} = -0.10$

$R_{IH} = -0.125$ $\quad R_{IH} = -0.375$

$R_{ED} = -0.10$ $\quad R_{EH} = -0.15$

$R_{EB} = -0.15$ $\quad R_{EF} = -0.10$

$R_{FE} = -0.125$ $\quad R_{FI} = -0.1875$ $\quad R_{FC} = -0.1875$

The rotation factors are recorded at the corresponding ends of the members in the computational scheme (Fig. 5.3).

Rotation Contributions

The contribution to the end moment M_{jm} by the rotation θ_j at joint j is given by

$$M'_{jm} = R_{jm}(M_j + \sum_m M'_{mj})$$

The calculation of the rotation contribution may be started at any joint and continued at other joints in any chosen sequence. The sequence adopted here is GHI–DEF.

First Cycle

(a) Joint G Since the joints H and D are initially locked, the contributions to the end moments at G are zero. Thus, the initial values of these joint moments are set to zero or $M'_{HG} = M'_{DG} = 0$ and the rotation contributions at G are

$$M'_{GH} = -0.125\ (+6.0 + 0.0 + 0.0) = -0.75 \text{ kN m}$$

$$M'_{GD} = -0.375\ (+6.0 + 0.0 + 0.0) = -2.25 \text{ kN m}$$

These rotation contributions are entered at the joint G below the respective fixed-end moment as shown in Fig. 5.3.

(b) Joint H At this joint again, $M'_{EH} = M'_{IH} = 0.0$. But, $M'_{GH} = -0.75$, as computed above at joint G. Therefore

$$M'_{HG} = -0.10(0.0 - 0.75 + 0.0 + 0.0) = +0.08 \text{ kN m}$$

Similarly

$$M'_{HE} = -0.3(0.0 - 0.75 + 0.0 + 0.0) = +0.23 \text{ kN m}$$

$$M'_{HI} = -0.1(-0.75) = +0.08 \text{ kN m}$$

In the same manner the calculations are performed at the other joints until the first cycle is completed.

Second Cycle

The results obtained from the first cycle are used to obtain better approximations to the rotation contributions. For example, consider the rotation contribution at joint D:

$$M'_{DG} = -0.187(+9.0 - 1.80 + 0.06) = -1.36 \text{ kN m}$$

$$M'_{DE} = -0.125(9.0 - 1.80 + 0.06) = -0.91 \text{ kN m}$$

$$M'_{DA} = -0.188(9.0 - 1.80 + 0.06) = -1.36 \text{ kN m}$$

A similar calculation is performed at all joints until the second cycle is completed. This procedure is performed until the two successive cycles furnish values differing by only a small acceptable amount. In this particular example, four cycle operations were considered adequate and the complete calculation scheme is shown in Fig. 5.3.

Final End Moments

The final end moments are determined from the relation

$$M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj}$$

Substituting the fixed-end moment values and the rotation contributions obtained from the scheme of calculation:

$$M_{AD} = 0.0 + 2(0.0) - 1.36 = -1.36 \text{ kN m}$$

$$M_{DA} = 0.0 + 2(-1.36) + 0.0 = -2.72 \text{ kN m}$$

$$M_{DE} = +9.0 + 2(-0.91) + 0.0 = +7.18 \text{ kN m}$$

$$M_{DG} = 0.0 + 2(-1.36) - 1.74 = -4.46 \text{ kN m}$$

$M_{ED} = -9.0 + 2(0.0) - 0.91 = -9.91$ kN m

$M_{GD} = 0.0 + 2(-1.74) - 1.36 = -4.84$ kN m

$M_{GH} = +6.0 + 2(-0.58) + 0.0 = +4.84$ kN m

$M_{HG} = -6 + 2(0.0) - 0.58 = -6.58$ kN m

$M_{EH} = M_{HE} = M_{EB} = M_{BE} = 0$

Notice that the final end moments to the right of the structural axis are equal in magnitude but opposite in direction due to the symmetry of both the loading system and the structure itself.

5.3 FRAMES WITH SIDESWAY

When a frame is either structurally unsymmetric or is symmetric with unsymmetrical loading, joint translation or sidesway occurs. Figure 5.4 shows member j–m of a frame with lateral displacement. The rotations at joints j and m are θ_{jm} and θ_{mj} respectively, and Δ_{jm} is the relative lateral displacement between j and m.

The slope deflection equation for the member j–m is

$$M_{jm} = M_{jm}^F - 2EK_{jm}(2\theta_j + \theta_m - 2\Delta_{jm}/L_{jm}) \quad [5.7]$$

or

$$M_{jm} = M_{jm}^F + 2M'_{jm} + M'_{mj} + M''_{jm} \quad [5.8]$$

where

$$M'_{jm} = -2EK_{jm}\theta_j$$

$$M'_{mj} = -2EK_{jm}\theta_m$$

$$M''_{jm} = \frac{6EK_{jm}\Delta_{jm}}{L_{jm}}$$

The symbols M'_{jm} and M'_{mj}, respectively, define the *rotation contributions* of the joints j and m to the total moment M_{jm}. In a similar manner, M''_{jm} constitutes the contribution of M_{jm} by the displacement Δ_{jm} and is therefore defined as the *displacement contribution*.

The algebraic sum of the end moments of all members meeting at joint j is zero.

$$\sum_m M_{jm} = 0$$

or

$$\sum_m M_{jm}^F + \sum_m (2M'_{jm} + M'_{mj} + M''_{jm}) = 0 \quad [5.9]$$

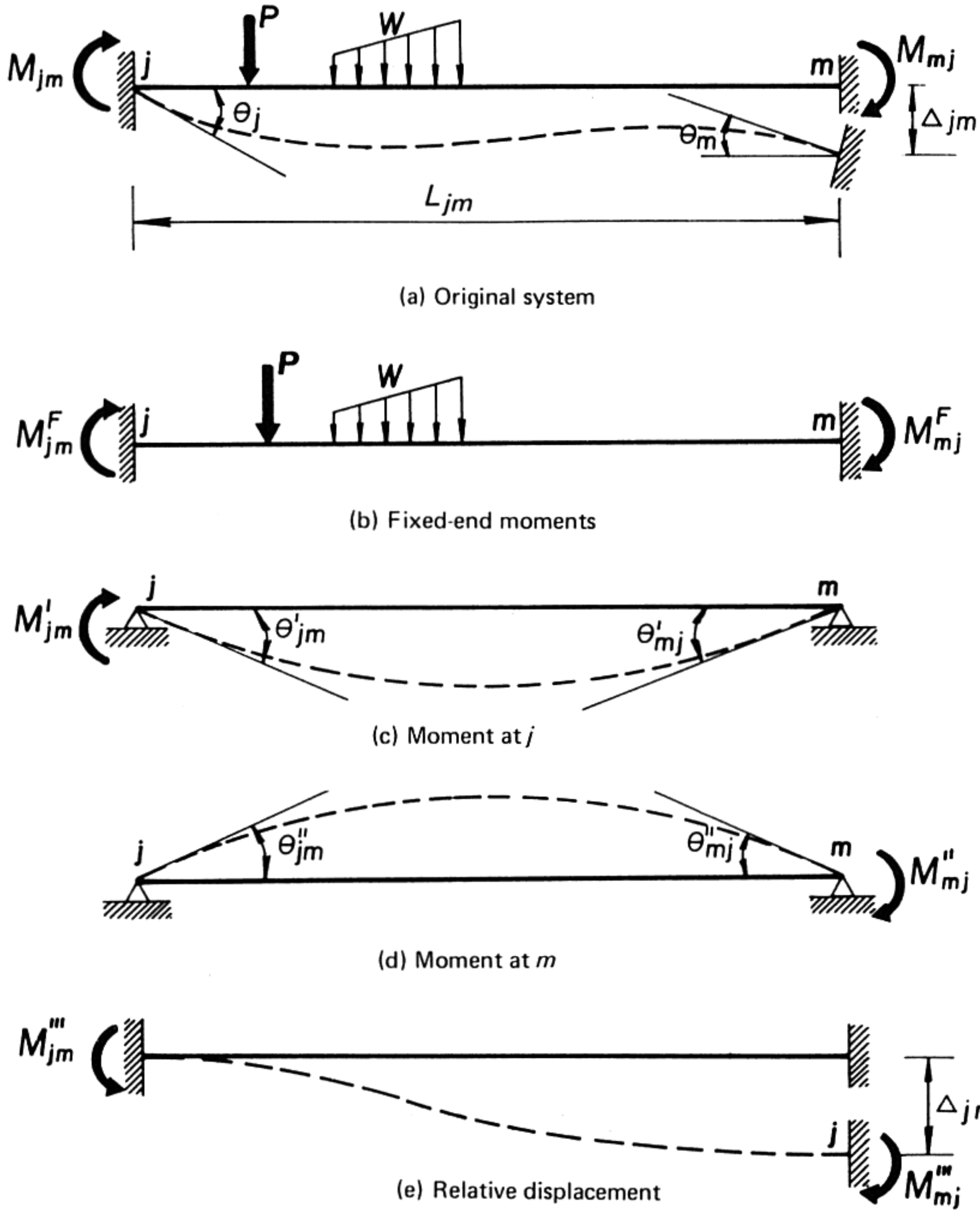


Figure 5.4

Defining the algebraic sum of the fixed-end moments at joint j as the *restraint moment*,

$$M_j = \sum_m M_{jm}^F$$

then [5.9] may be written as

$$\sum_m M'_{jm} = -\frac{1}{2} M_j + \sum_m (M'_{mj} + M''_{jm}) \qquad [5.10]$$

Equation [5.10] forms the basis for Kani's method for frames with sidesway where the rotation and displacement contributions must be evaluated for member j–m. These contributions and the fixed-end moments are added algebraically to determine the final joint moment M_{jm} (see [5.8]). However, while the evaluation of the rotation contributions are easily computed as described in Section 5.2, the determination of the displacement contribution M''_{jm} could be involved depending on the type of loading on the structure and whether there are columns of different length. The cases arising in the determination of displacement contributions are discussed below.

5.3.1 Vertical Loading

Consider the frame shown in Fig. 5.5(a) subjected to vertical loading only. The frame undergoes sidesway due to the unsymmetrically placed vertical loads. The frame analysis may be carried out in two steps as was done in the Cross moment distribution (Section 4.6).

(a) No-sway Solution

As shown in Fig. 5.5(b), artificial joint restraints are applied at storey heights to prevent sidesway. The analysis of the frame without sidesway follows the same procedure presented in Section 5.3. After the joint moments are evaluated the artificial joint restraints may be determined if required, from equilibrium considerations of the shear forces at every storey.

(b) Sway Solution

Since the artificial joint restraints introduced in the *no-sway solution* do not actually exist in the given frame, their presence may be nullified by applying a consistent force system (Fig. 5.5(c)) whose forces are equal in magnitude but opposite in direction to the respective artificial joint restraints.

By cutting horizontally through all columns at the rth storey and from the consideration of equilibrium conditions, the algebraic sum of the column shear must be zero,

$$\sum_r V_{jm} = 0$$

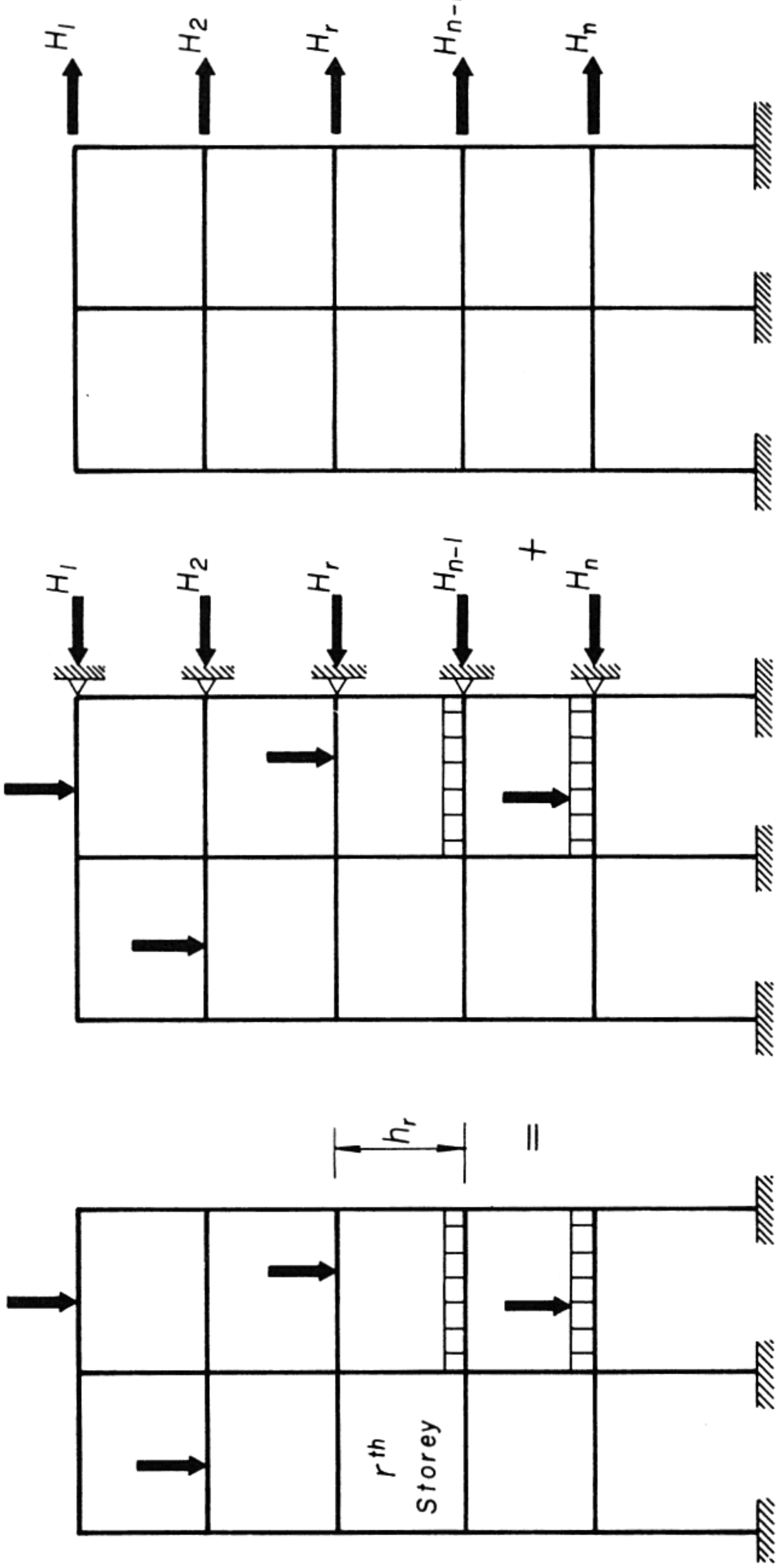
where V_{jm} = shear in column j–m of the rth storey. Let h_r represent the column height of the rth storey, then

$$V_{jm} = \frac{M_{jm} + M_{mj}}{h_r} \qquad [5.11]$$

But

$$M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj} + M''_{jm}$$

$$M_{mj} = M^F_{mj} + 2M'_{mj} + M'_{jm} + M''_{jm}$$



(a) Loaded frame

(b) No – Sway

(c) Consistent lateral force system

Figure 5.5

Since vertical loading only is considered, there are no intermediate loads on the column $j\text{–}m$ and the fixed-end moments are zero. Thus

$$M''_{jm} = M''_{mj} = \frac{6EK_{jm}\Delta_{jm}}{L_{jm}} \qquad [5.12]$$

Substituting into [5.11],

$$V_{jm} = \frac{1}{h_r}(3M'_{jm} + 3M'_{mj} + 2M''_{jm}) \qquad [5.13]$$

Taking the sum of the column shears at the rth storey,

$$\sum_r V_{jm} = \sum_r \frac{1}{h_r}[3(M'_{jm} + M'_{mj}) + 2M''_{jm}] = 0 \qquad [5.14]$$

Solving for the sum of the displacement contribution,

$$\sum_r M''_{jm} = -\frac{3}{2}\sum_r (M'_{jm} + M'_{mj})$$

Hence, the algebraic sum of the displacement contribution of all columns at the rth storey is determined to be -1.5 times the sum of the rotation contributions of the column ends of the same storey. Since all the columns at this storey undergo the same lateral displacement, the displacement contribution M''_{jm} for any individual column is proportional to its relative stiffness value (see [5.12]). The displacement contribution of any column is, therefore, obtained by distributing the sum $\sum_r M''_{jm}$ among the columns in the rth storey in proportion to their stiffness values. Thus, for any column $j\text{–}m$

$$M''_{jm} = -1.15\left(\frac{K_{jm}}{\sum_r K_{jm}}\right)\sum_r (M'_{jm} + M'_{mj}) \qquad [5.15]$$

where $\sum_r K_{jm}$ = the sum of the K values of all columns of the rth storey.

In order to make the computation more convenient, and to make an analogy to rotation factor, a *displacement factor* D_{jm} is defined for column $j\text{–}m$ as

$$D_{jm} = -1.5\left(\frac{K_{jm}}{\sum_r K_{jm}}\right) \qquad [5.16]$$

Equation [5.15] may be written as

$$M''_{jm} = D_{jm}\sum_r (M'_{jm} + M'_{mj}) \qquad [5.17]$$

Since the displacement contribution M''_{jm} and the rotation distribution M'_{jm} and M'_{mj} are interrelated, [5.17] may be solved by using the same convenient Gauss–Seidel iteration scheme. Once the displacement contributions are known the rotation contributions M'_{jm} are obtained from [5.6] and [5.9] as

$$M'_{jm} = R_{jm}[M_j + \sum_r (M'_{mj} + M''_{jm})] \qquad [5.18]$$

and from these, again, the following approximation of the displacement and rotation contributions are calculated until the results of a desired accuracy are obtained.

The final end moments are then obtained by using [5.8].

EXAMPLE 5.2 *Find the joint moments of the symmetric frame subjected to unsymmetric vertical loading shown in Fig. 5.6.*

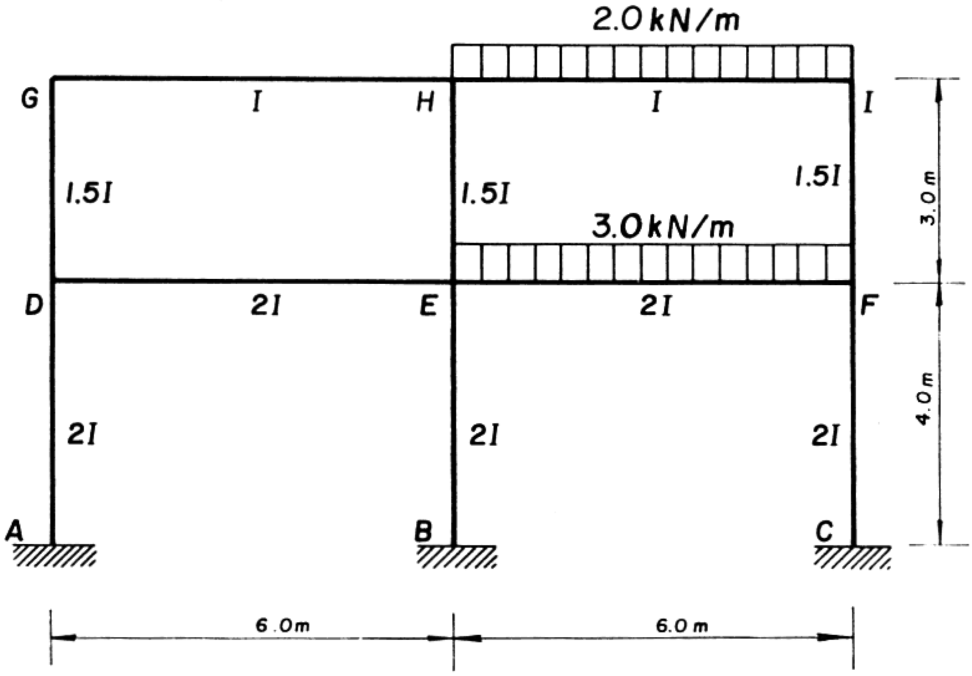


Figure 5.6

Fixed-End Moments

$$M^F_{EF} = -M^F_{FE} = \frac{3(6)^2}{12} = 9.0 \text{ kN m}$$

$$M^F_{HI} = -M^F_{IH} = \frac{2(6)^2}{12} = 6.0 \text{ kN m}$$

Restraint Moments ($M_j = \sum_m M^F_{jm}$)

$$M_G = M_D = 0$$

$$M_H = -M_I = 6.0 \text{ kN m}$$

$$M_E = -M_F = 9.0 \text{ kN m}$$

The relative stiffness values and the rotation factors have the same values as in Example 5.1.

Displacement Factors $(D_{jm} = -1.5K_{jm}/\sum_{m} K_{jm})$

$$D_{AD} = D_{BE} = D_{CF} = -1.5\left(\frac{0.5}{1.5}\right) = -0.5$$

$$D_{DG} = D_{EH} = D_{FI} = -1.5\left(\frac{0.5}{1.5}\right) = -0.5$$

Notice that the restraint moments M_j and the rotation factors R_{jm} are recorded in the usual manner (Example 5.1) in the computational scheme shown in Fig. 5.7. The displacement factors D_{jm} are also entered at the centre of the relevant columns in the same computational scheme.

First Cycle

The rotation contributions are first computed which are then used to evaluate the displacement contributions.

(a) Rotation Contributions, $M'_{jm} = R_{jm}[M_j + \sum_{m}(M'_{mj} + M''_{jm})]$

Since M'_{mj} and M''_{jm} are initially known, they are set to zero at the first cycle.

(i) Joint G:

$$\text{Set } M'_{DG} = M'_{HG} = M''_{GD} = 0.0$$
$$M_G = 0.0$$

Thus

$$M'_{GD} = M'_{GH} = -0.375(M_G + M'_{DG} + M'_{HG} + M''_{DG}) = 0.0$$

(ii) Joint H:

$$\text{Set } M'_{EH} = M'_{FH} = M''_{EH} = 0.0$$
$$M'_{GH} = 0.0 \qquad \text{(as found in (i))}$$
$$M_H = +6.00$$

Thus

$$M'_{HG} = -0.100(6.0 + 0.0 + 0.0 + 0.0) = -0.60 \text{ kN m}$$

Similarly

$$M'_{HE} = -0.300(6.0 + 0.0 + 0.0 + 0.0) = -1.80 \text{ kN m}$$

$$M'_{HI} = -0.100(6.0 + 0.0 + 0.0 + 0.0) = -0.60 \text{ kN m}$$

In the same manner all rotation contributions are entered below the relevant fixed-end moments.

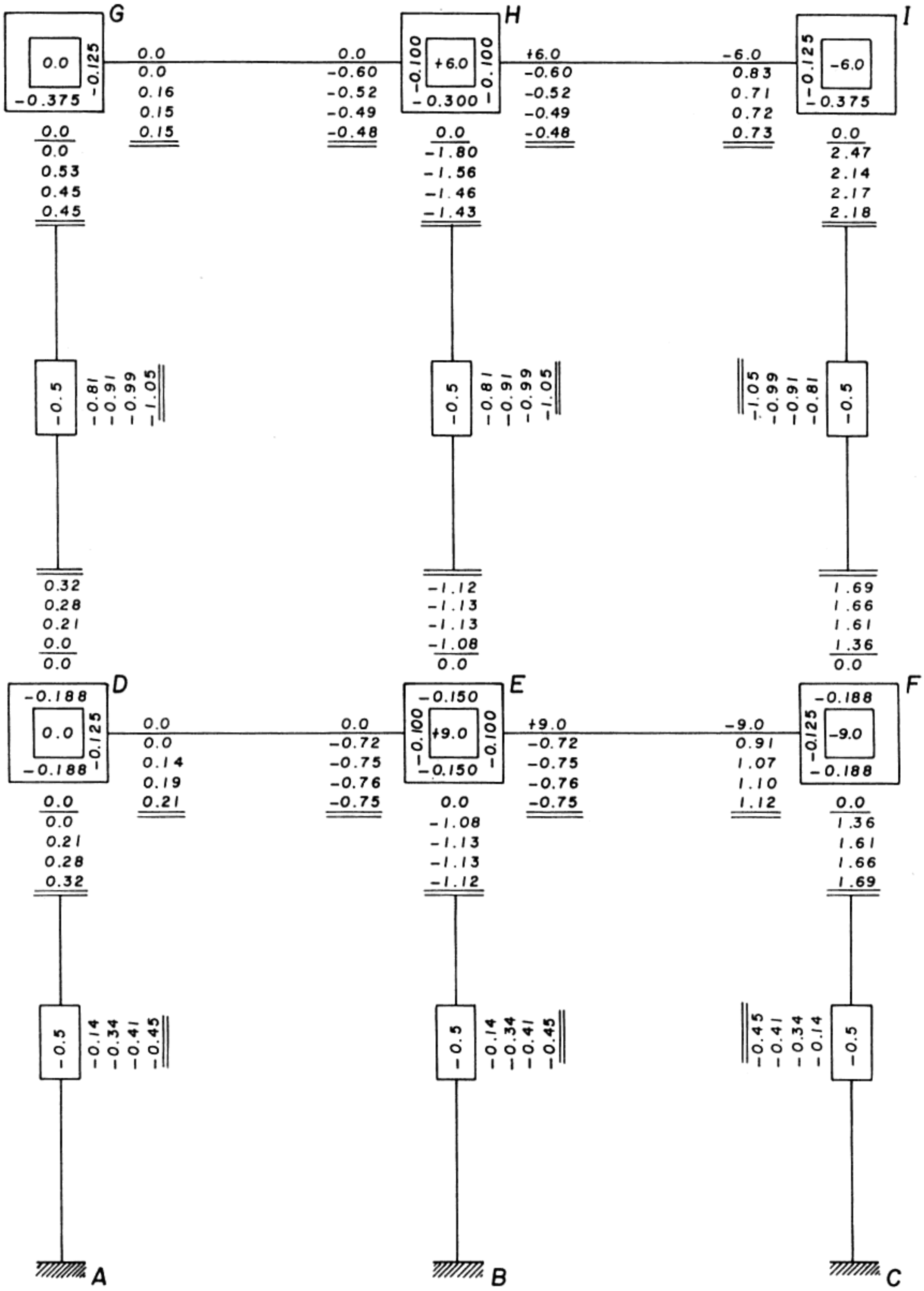


Figure 5.7

(b) *Displacement Contributions*, $M''_{jm} = D_{jm} \sum_r (M'_{jm} + M'_{mj})$

The displacement contributions are computed by multiplying the algebraic term of the rotation contributions of all columns of a storey with the displacement factors of the individual columns.

Thus

$$M''_{DG} = M''_{EH} + M''_{FI} = -0.50(0.0 + 0.0 - 1.80 - 1.08 + 2.48 + 1.36)$$
$$= -0.48 \text{ kN m}$$

Similarly

$$M''_{AD} = M''_{BE} = M''_{CF} = -0.50(0.0 + 0.0 - 1.08 + 0.0 + 1.36 + 0.0)$$
$$= -0.14 \text{ kN m}$$

The displacement contributions are recorded at the middle of the relevant columns as shown in the scheme of calculation (Fig. 5.7).

Second Cycle

The results obtained from the first cycle are used to obtain better approximations to the rotation and displacement contributions.

(a) Rotation Contributions

(i) Joint G:

$$M'_{GD} = -0.375(M_G + M'_{DG} + M'_{HG} + M''_{DG})$$
$$= -0.375(0.00 + 0.00 - 0.60 - 0.48) = 0.41 \text{ kN m}$$

Similarly

$$M'_{GH} = -0.125(0.00 + 0.00 - 0.60 - 0.48) = 0.14 \text{ kN m}$$

(ii) Joint H:

$$M'_{GH} = +0.14 \text{ kN m} \quad \text{(as in (i) above)}$$
$$M''_{EH} = -1.08 \text{ kN m}$$
$$M'_{IH} = +0.83 \text{ kN m}$$
$$M''_{EH} = -0.48 \text{ kN m}$$
$$M_H = +6.0 \text{ kN m}$$

Therefore

$$M'_{HG} = M'_{HI} = -0.100(6.0 + 0.14 - 1.08 + 0.90 - 0.81)$$
$$= -0.54 \text{ kN m}$$
$$M'_{HE} = -0.300(6.0 + 0.18 - 1.08 + 0.83 - 0.48)$$
$$= -1.62 \text{ kN m}$$