Similarly, all rotation contributions of the second cycle are entered below the relevant fixed-end moments.

(b) Displacement Contributions

$$M''_{DG} = M''_{EH} = M''_{FI} = -0.50(0.41 + 0.17 - 1.62 - 1.17 + 2.12 + 1.56)$$
$$= -0.74 \text{ kN m}$$
$$M''_{AD} = M''_{BE} = M''_{CF} = -0.50(0.17 + 0.00 - 1.17 + 0.00 + 1.56 + 0.00)$$
$$= -0.28 \text{ kN m}$$

This completes the second cycle operations. The procedure is then repeated until two successive cycles furnish sets of values differing by a very small acceptable amount. In this particular example, the scheme of computation shows four cycles to be sufficient (Fig. 5.7).

Final End-Moments, $M_{jm} = M^F_{jm} + 2M''_{jm} + M'_{mj} + M''_{jm}$

$$M_{AD} = 0.0 + 2(0.0) + 0.33 - 0.45 = -0.12 \text{ kN m}$$
$$M_{DA} = 0.0 + 2(0.33) + 0.0 - 0.45 = 0.21 \text{ kN m}$$
$$M_{DE} = 0.0 + 2(0.22) + (-0.75) = 0.31 \text{ kN m}$$
$$M_{FE} = -9.0 + 2(1.12) + (-0.75) = -7.51 \text{ kN m}$$

and so on.

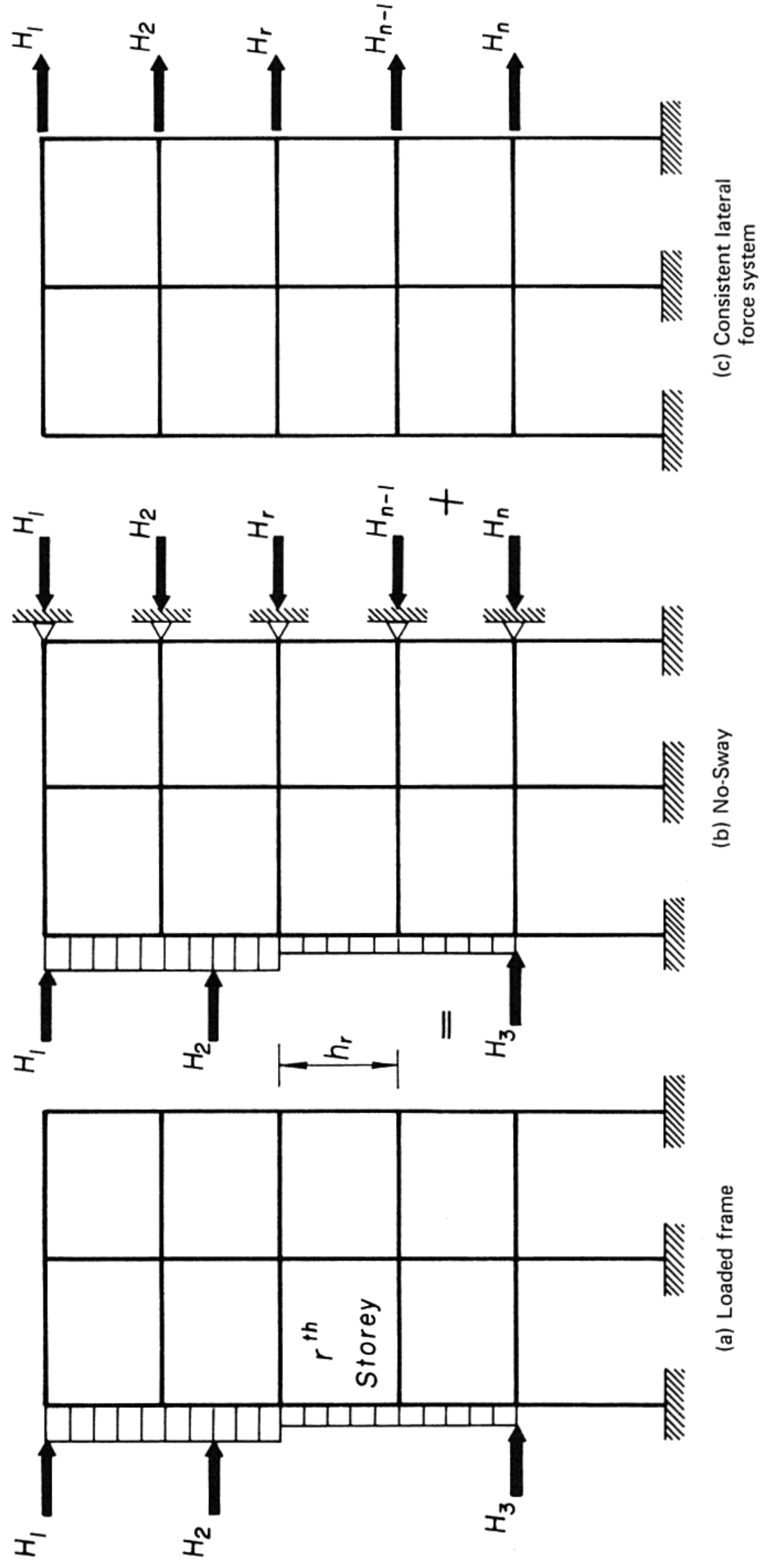
5.3.2 Horizontal Loading

The principle in determining the displacement contributions in frames subjected to horizontal loading remain the same as in vertical loading. However, the presence of horizontal loads on the frame requires an additional effort in the computation.

Consider a frame subjected to horizontal loads applied as shown in Fig. 5.8(a). Again, the analysis may be carried out in two steps:

(a) No-Sway Solution

Artificial joint restraints are applied at storey heights as shown in Fig. 5.8(b) to prevent sidesway. These joint restraints may be determined from no-sway solution (section 5.3).



(a) Loaded frame

(b) No-Sway

(c) Consistent lateral force system

Figure 5.8

(b) Sway Solution

Since the artificial joint restraints do not actually exist, they may be eliminated by applying a consistent force system (Fig. 5.8(c)) whose storey shear at any section is equal in magnitude but opposite in direction to the algebraic sum of the applied horizontal force above that section. Designating the sum of the restraint forces above the rth storey as the storey shear V_r, the horizontal equilibrium condition above the rth storey requires that

$$V_r = H_1 + H_2 + \ldots H_r = \sum_r H$$

$$= \sum_r \frac{1}{h_r}(M_{jm} + M_{mj}) \qquad [5.19]$$

Assuming the columns are not subjected to intermediate horizontal loads (such as by applying equivalent loads at the storey heights that will give the same global effect on the frame), all fixed-end moments become zero. Using [5.8] to expand the end-moments M_{jm} and M_{mj}, [5.19] may be written as

$$V_r = \frac{1}{h_r}\sum_r [3(M'_{jm} + M'_{mj}) + 2M''_{jm}]$$

Rearranging:

$$\sum_r M''_{jm} = -\frac{3}{2}\left[-\frac{V_r h_r}{3} + \sum_r (M'_{jm} + M'_{mj})\right] \qquad [5.20]$$

The quantity $V_r h_r/3$, which is one-third of the product of the storey shear and storey height, is defined as the *storey moment* M'_r.

$$M_r = -\frac{V_r h_r}{3}$$

Equation [5.20] gives the sum of the displacement contributions of all columns in the rth storey. As explained in the case of vertical loads, the moment in any column j–m is obtained by distributing this sum in proportion to their K values. Thus

$$M''_{jm} = D_{jm}[M_r + \sum_r (M'_{jm} + M'_{mj})] \qquad [5.21]$$

Notice that the displacement factors D_{jm} are the same as for vertical loading, so that the displacement contribution in the case of horizontal loading (see [5.21]) differs in the extra term M_r from the case of vertical loading (see [5.17]). Therefore, the analysis of frames subjected to horizontal loading differs from the analysis of frames with vertical loading only by the extra term M_r which must be calculated for each storey and be added algebraically to the sum of the rotation contributions of the two ends of the columns of the storey considered.

EXAMPLE 5.3 *Find the joint moments of the frame subjected to horizontal loads as shown in Fig. 5.9.*

The relative stiffness values, rotation factors, and displacement factors are the same as in Example 5.2 and are recorded in the computational schemes as usual (Fig. 5.9).

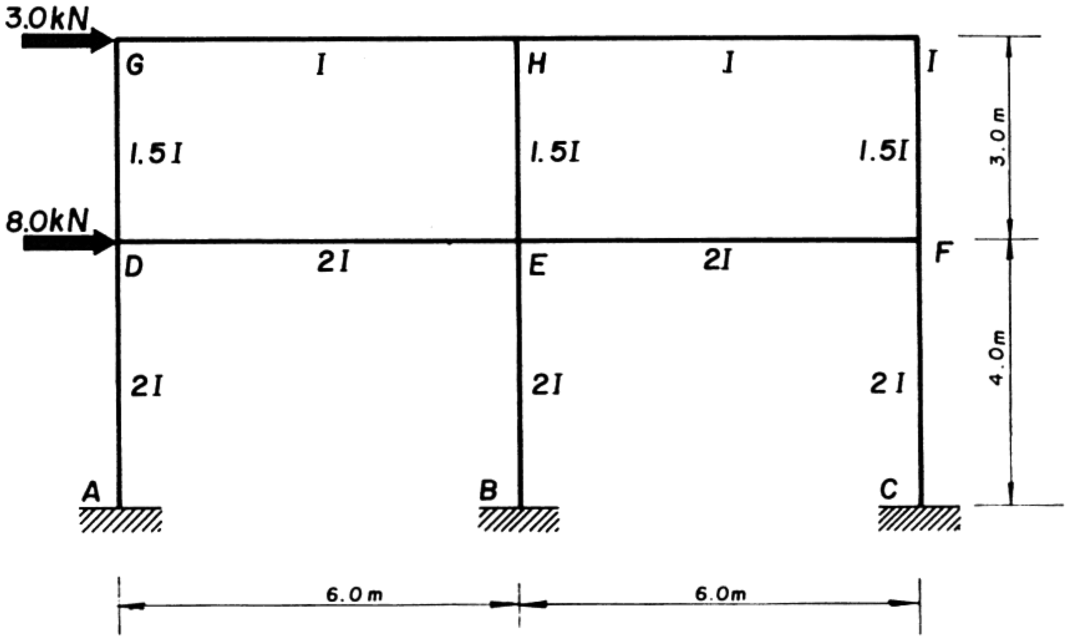


Figure 5.9

Storey Shears and Storey Moments

(i) Second storey

Storey shear, $V_r = 3.0$ kN

$$\text{Storey moment}, M_r = -\frac{V_r h_r}{3} = -\frac{3.0(3.0)}{3} = -3.0 \text{ kN m}$$

(ii) First storey

Storey shear, $V_r = 3.0 + 8.0 = 11.0$ kN

$$\text{Storey moment}, M_r = -\frac{11.0(4.0)}{3} = -14.67 \text{ kN m}$$

The storey moments are recorded in the computational scheme (Fig. 5.10) at the centre of the relevant storey.

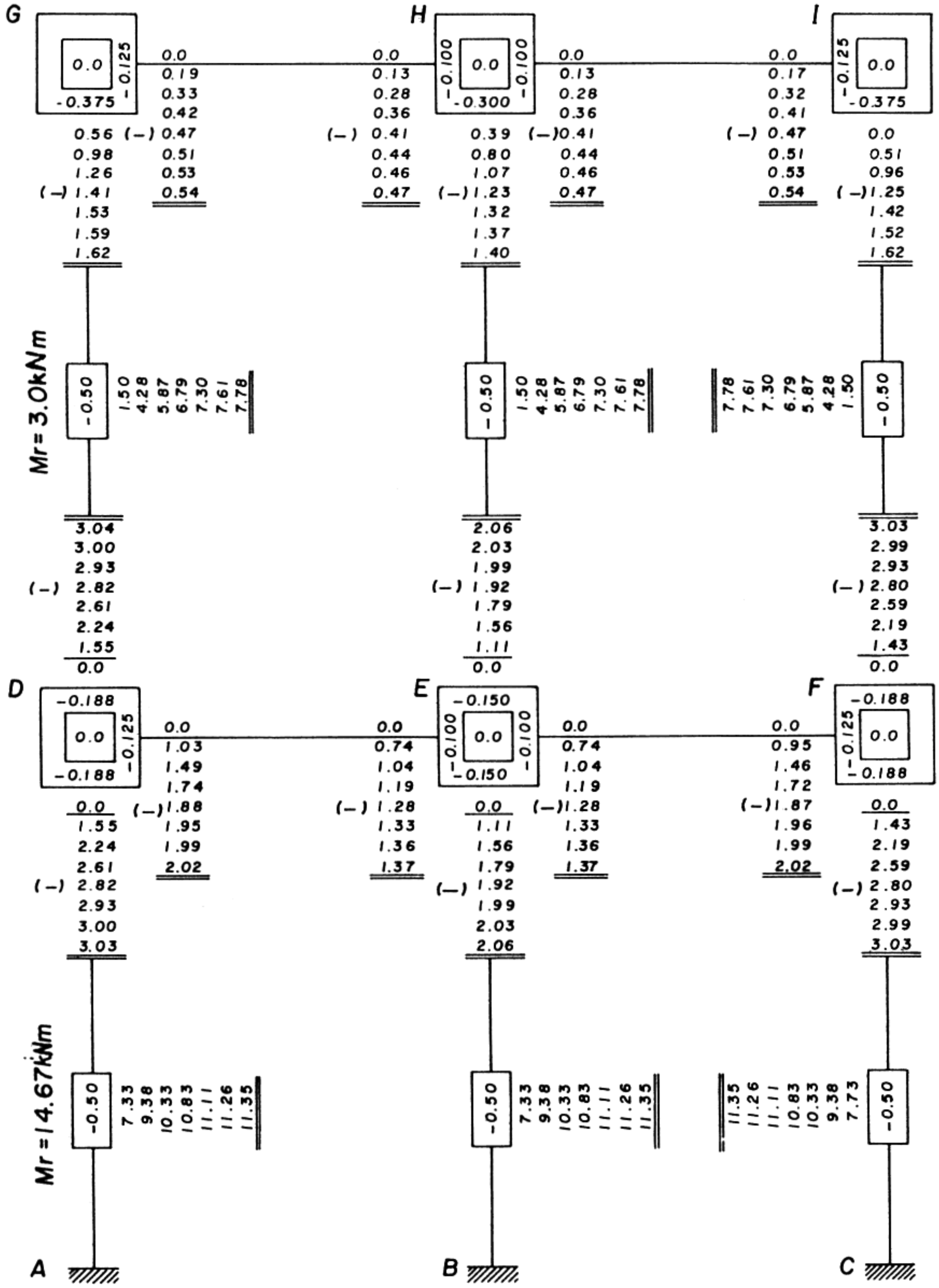


Figure 5.10

Displacement and Rotation Contributions

In frames subjected to horizontal loads, the displacement contributions are usually significantly larger than the rotation contributions. Hence, the displacement contributions are calculated first as illustrated in the following calculations.

First Cycle

(a) Displacement Contribution, $M''_{jm} = D_{jm}[M_r + \sum_r (M'_{jm} + M'_{mj})]$

Since the rotation contributions are initially zero, the displacement contributions are

$$M''_{DG} = M''_{EH} = M''_{FI} = -0.5(-3.0 + 0.0 + 0.0)$$
$$= +1.50 \text{ kN m}$$
$$M''_{AD} = M''_{BE} + M''_{CF} = -0.5(-14.67 + 0.0 + 0.0)$$
$$= +7.33 \text{ kN m}$$

(b) Rotation Contributions, $M'_{jm} = R_{jm}[M'_{mj} + \sum_m (M'_{mj} + M''_{jm})]$

(i) Joint G:

At this joint $M_G = M'_{HG} = M'_{DG} = 0$

and $M''_{GD} = -1.50$ kN m

$$M'_{GD} = -0.375(0.0 + 0.0 + 1.50) = -0.56 \text{ kN m}$$
$$M_{GH} = -0.375(0.0 + 0.0 + 1.50) = -0.19 \text{ kN m}$$

(ii) Joint H:

At this joint, $M_H = M'_{EH} = M'_{IH} = 0$

$M'_{GH} = -0.19$ kN m

and $M''_{EH} = +1.50$ kN m

Thus

$$M'_{HG} = -0.100(0.0 - 0.19 + 0.0 + 0.0 + 1.50) = -0.13 \text{ kN m}$$

Similarly

$$M'_{HE} = -0.300(0.0 - 0.19 + 0.0 + 0.0 + 1.50) = -0.39 \text{ kN m}$$
$$M'_{HI} = -0.100(0.0 - 0.19 + 0.0 + 0.0 + 1.50) = -1.3 \text{ kN m}$$

In the same manner all the rotation contributions are calculated until the first cycle is completed.

Second Cycle

(a) Displacement Contributions

The displacement contributions are obtained by using [5.21] where the results from the first cycle are used to obtain approximations.

$$M''_{\mathrm{DG}} = M''_{\mathrm{EH}} = M''_{\mathrm{FI}} = -0.5(-3.0 - 0.50 - 1.55 - 0.39 - 1.11 - 0.51 - 1.43)$$
$$= +4.28 \text{ kN m}$$

$$M''_{\mathrm{AD}} = M''_{\mathrm{BE}} = M''_{\mathrm{CF}} = -0.50(-14.67 - 1.55 - 1.11 - 1.43)$$
$$= +9.38 \text{ kN m}$$

(b) Rotation Contributions

There are no particular points to be noted here and similar calculations are performed until the second cycle is completed. The procedure is repeated until two successive cycles furnish sets of values differing by a very small acceptable amount. In this particular example, the scheme of computation shows seven cycles to be sufficient (Fig. 5.10).

Final End Moments, $M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj} + M''_{jm}$

$$M_{\mathrm{AD}} = 0.0 + 2(0.0) - 3.03 - 11.35 = +8.32 \text{ kN m}$$
$$M_{\mathrm{DA}} = 0.0 + 2(-3.04) + 0.0 - 11.35 = +5.29 \text{ kN m}$$
$$M_{\mathrm{DE}} = 0.0 + 2(-2.02) - 1.37 = -5.41 \text{ kN m}$$
$$M_{\mathrm{DG}} = 0.0 + 2(-3.04) - 1.62 + 7.78 = +0.10 \text{ kN m}$$
$$M_{\mathrm{GD}} = 0.0 + 2(-1.62) - 3.03 + 7.78 = +1.51 \text{ kN m}$$
$$M_{\mathrm{GH}} = 0.0 + 2(-0.54) - 0.47 = -1.55 \text{ kN m}$$

and so on.

5.3.3 Frames With Columns of Unequal Heights

For a frame with a storey containing columns of unequal heights, the calculations of the rotation contributions in all storeys remain the same as described earlier. Also, the computation of the displacement contributions for those storeys with equal heights are not altered. However, in establishing the governing equations for the calculation of the displacement contribution for the storey with unequal heights, supplementary consideration is needed.

Consider the frame shown in Fig. 5.11. In the storey which has columns of unequal heights, an ***arbitrary*** column that appears most frequently is taken as the storey height. Let

h_r = ***storey height*** in the *r*th storey which has columns of unequal heights

h_{jm} = height of any other column j–m in the *r*th storey

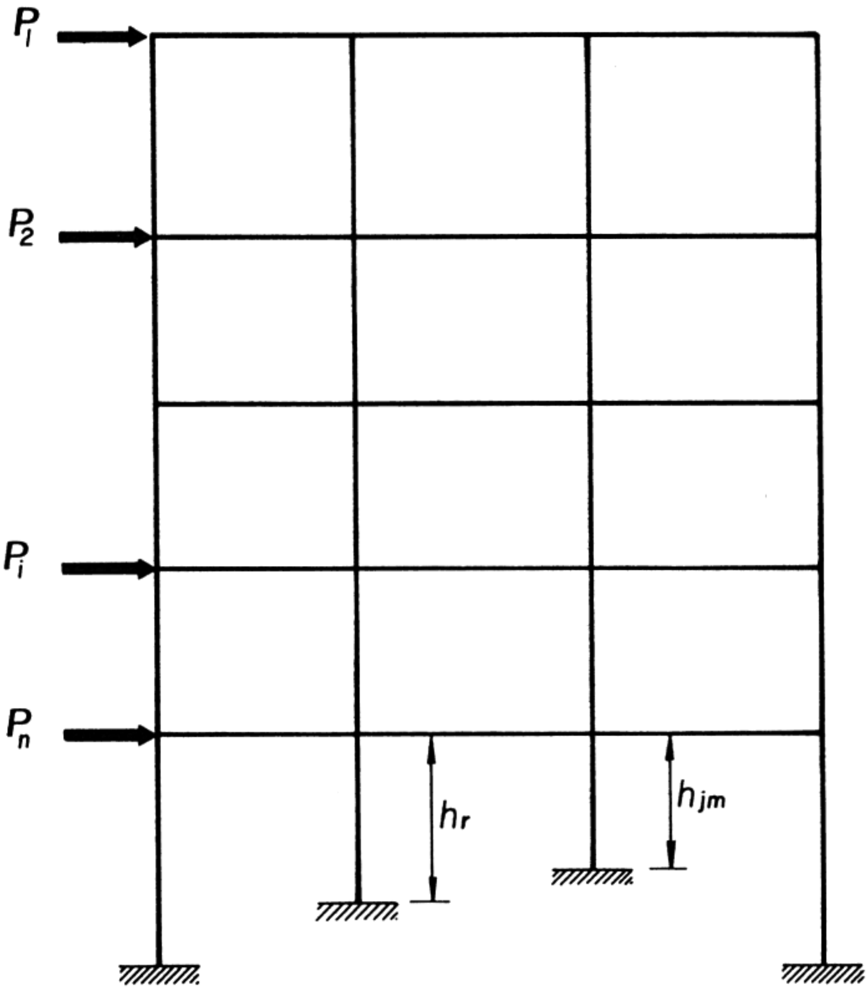


Figure 5.11

Writing the equilibrium condition of the horizontal forces at the *r*th storey

$$V_r + \sum_r V_{jm} = 0 \tag{5.22}$$

or

$$V_r + \sum_r \frac{1}{h_{jm}} (M_{jm} + M_{mj})$$

$$V_r h_r + \sum_r \frac{h_r}{h_{jm}} (M_{jm} + M_{mj}) = 0$$

Introducing a factor defined as the *height reduction* factor for the j–m column in the rth storey,

$$C_{jm} = \frac{h_r}{h_{jm}} \qquad [5.23]$$

Equation [5.22] is written as

$$Q_r h_r + \sum_r C_{jm}(M_{jm} + M_{mj}) = 0 \qquad [5.24]$$

Substituting the values of M_{jm} and M_{mj} given by

$$M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj} + M''_{jm}$$

$$M_{mj} = M^F_{mj} + 2M'_{mj} + M'_{jm} + M''_{jm}$$

into the shear equations and taking into consideration that the fixed-end moments are zero,

$$V_r h_r + \sum_r C_{jm}(3M'_{jm} + 3M'_{mj} + 2M''_{jm}) = 0$$

Therefore

$$\sum_r C_{jm} M''_{jm} = -1.5(M_r + \sum_r C_{jm}(M'_{jm} + M'_{mj}) \qquad [5.25]$$

where

$$M_r = \frac{V_r h_r}{3} = \text{storey moment}$$

Since

$$M''_{jm} = \frac{6EK_{jm}\Delta_{jm}}{h_{jm}}$$

them M''_{jm} is proportional to K_{jm}/h_{jm} and also to $C_{jm}K_{jm}$. Also, since Δ_{jm} is the same for all columns of the storey under consideration,

$$\frac{M''_{jm}}{\sum_r C_{jm} M_{jm}} = \frac{C_{jm}K_{jm}}{\sum_r C^2_{jm}K_{jm}} \qquad [5.26]$$

From [5.25] and [5.26], the basic equation for determining the displacement contribution M''_{jm} may be written as

$$M''_{jm} = D'_{jm}\,[M_r + \sum_r C_{jm}(M'_{jm} + M'_{mj})] \qquad [5.27]$$

where

$$D'_{jm} = -1.5\left(\frac{C_{jm}K_{jm}}{\sum_r C^2_{jm}K_{jm}}\right) = \textit{displacement factor}$$

For the storey with unequal column heights, the following changes must be noted:

(a) a *reduction factor* C_{jm} must be introduced;
(b) a *modified displacement factor* D'_{jm} must be used.

Having introduced these factors, [5.26] and [5.27] along with [5.8] and [5.18] are used to determine the end moments.

5.4 PROBLEMS

5.1 Solve problem 3.1 using Kani's method of moment distribution.

5.2 Solve problem 3.2 using Kani's method of moment distribution.

5.3 Solve problem 3.3 using Kani's method of moment distribution.

5.4 Solve the problem of Example 4.7 using Kani's method of moment distribution.

5.5 Solve the problem of Example 4.8 using Kani's method of moment distribution.

6. Influence Lines for Indeterminate Structures

6.1 INTRODUCTION

The determination of the maximum and sometimes the minimum structural effects of the appropriate load system is an important preliminary step in the analysis and design of structures. Structures subjected to moving or movable loads invariably involve the calculation of the maximum or minimum values of the bending moment, shear force, deflection and so forth, by preparing the influence lines for the various structural effects.

The influence lines for statically determinate structures may be drawn by connecting a few key ordinates with straight lines. However, influence lines for indeterminate structures are curved and therefore cannot be drawn so easily. The first step in preparing the influence lines for the various functional values is to determine the influence lines for the redundants. The next step then is to find the influence lines for any other reaction, moment or shear, etc. that can be computed by statics. The influence lines for different functions in statically indeterminate structures may be obtained using the Müller-Breslau principle backed by computational techniques such as the conjugate beam principle, Cross moment distribution, and energy methods.

6.2 STRUCTURES WITH A SINGLE REDUNDANT REACTION

The influence lines for indeterminate structures may be constructed by using either *statical* or *kinematic* methods. When using the statical method consideration of equilibrium alone is utilised. This may be demonstrated by considering a propped cantilevered beam of uniform cross-section. It is desired to prepare an influence line for the vertical reaction at support B shown in Fig. 6.1.

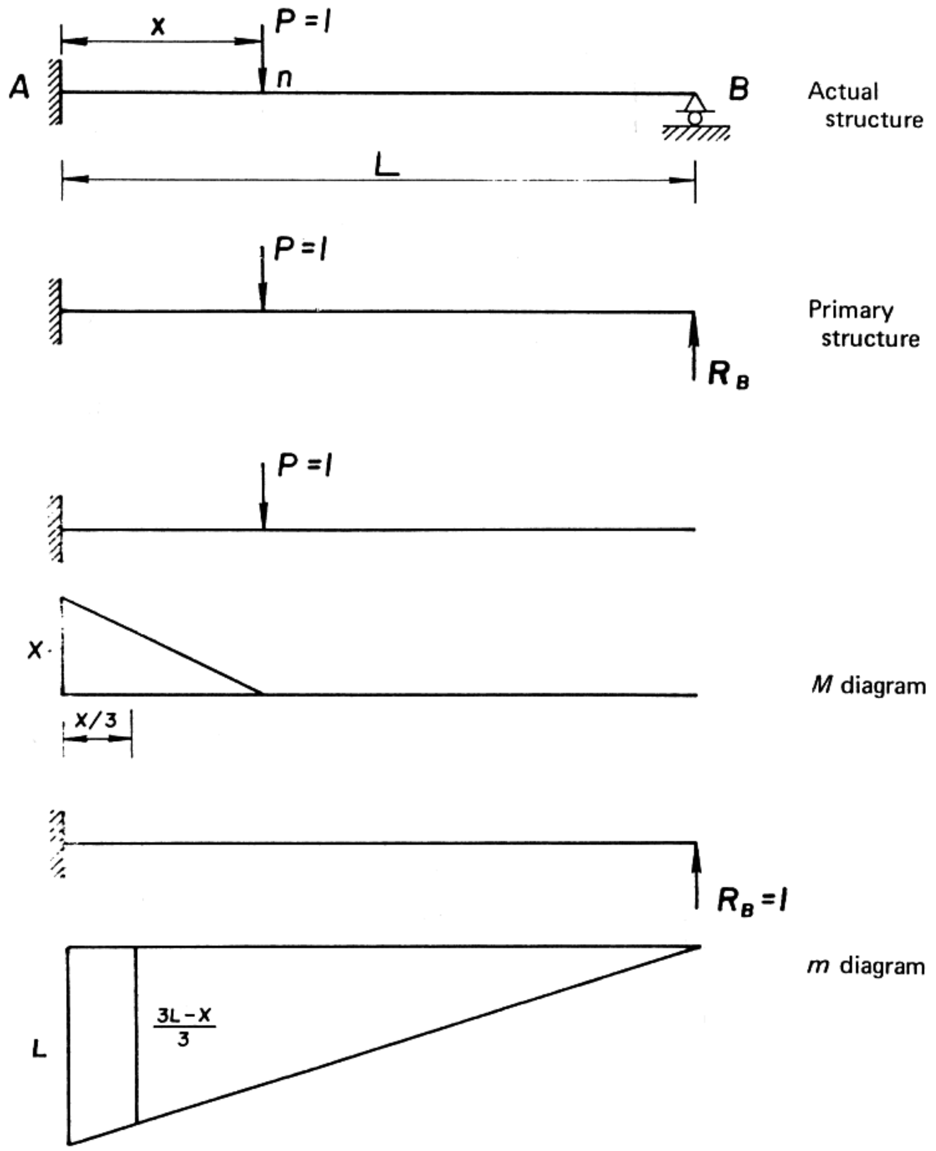


Figure 6.1

The reaction R_B is determined from the compatibility condition

$$R_B \delta_{bb} + \delta_{bo} = 0 \qquad [6.1]$$

from which

$$R_B = -\frac{\delta_{bo}}{\delta_{bb}} \qquad [6.2]$$

The deflections δ_{bo} and δ_{bb} are determined from the M and m diagrams (Fig. 6.1).

$$\delta_{bo} = \int \frac{Mmdx}{EI}$$

$$= -\frac{1}{EI}\left(\frac{x^2}{2}\right)\left(\frac{3L-x}{3}\right) = -\frac{x^2(3L-x)}{6EI}$$

and

$$\delta_{bb} = \int \frac{m^2 dx}{EI}$$

$$= \frac{1}{EI}\left(\frac{L^2}{2}\right)\left(\frac{2L}{3}\right) = \frac{L^3}{3EI}$$

The reaction at B is therefore

$$R_B = -\frac{\delta_{bo}}{\delta_{bb}} = \frac{x^2(3L - x)}{2L^3}$$

This equation gives the value of the reaction R_B when the unit load $P = 1$ is applied at any position along the beam which is therefore the influence line for the reaction at B. Using statics, the influence line for any other reaction, shear or moment may be determined.

Suppose it is required to find the influence line for the shear at the midspan of the beam. From statics, the following may be determined:

$$(V_C)_{\text{left}} = -R_B = -\frac{x^2(3L - x)}{2L^3} \text{ for } 0 < x < L/2$$

$$(V_C)_{\text{right}} = 1 - R_B = 1 - \frac{x^2(3L - x)}{2L^3} \text{ for } L/2 < x < L$$

The influence line for the bending moment at the midspan of the beam may also be evaluated in a similar manner.

The bending moment at midspan is

$$M_C = R_B \frac{L}{2}$$

$$= \frac{x^2(3L - x)}{4L^2} \text{ for } 0 < x < L/2$$

$$M_C = R_B \frac{L}{2} - 1\left(x - \frac{L}{2}\right)$$

$$= \frac{x^2(3L - x) - 4L^2x + 2L^3}{4L^2} \text{ for } L/2 < x < L$$

The ordinates of the influence line at quarter points for the reaction at B, R_B, the shear at midspan, V_C, and the bending moment at midspan, M_C, are given in Table 6.1 and also shown in Fig. 6.2.

To demonstrate the *kinematic* method of constructing influence lines consider the same propped cantilever beam of Fig. 6.1 where the reaction R_B is taken as redundant. Compatibility condition gives

$$R_B\delta_{bb} + \delta_{bo} = 0 \qquad [6.3]$$

Table 6.1 Ordinates of influence lines

x	R_B	V_C	M_C
0	0	0	0
$0.25L$	0.0859	−0.0859	$0.0430L$
$0.50L$	0.3125	−0.3125 or +0.6875	$0.1563L$
$0.75L$	0.6328	+0.2975	$0.0664L$
L	1.0	0	0

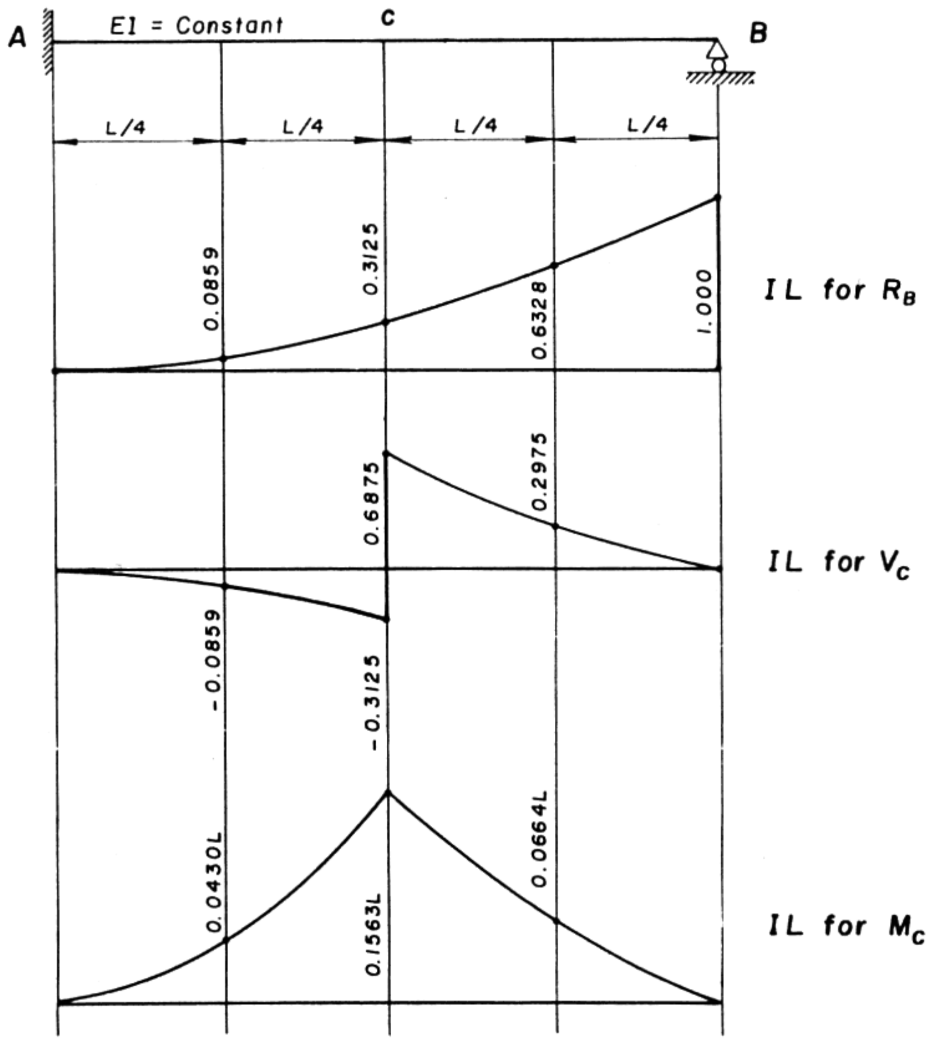


Figure 6.2

In this case the actual loading on the beam which produces the displacement δ_{bo} is the unit load $P = 1$ at the movable point n, so that the second term can be written as δ_{bn}. Hence

$$R_B = -\frac{\delta_{bn}}{\delta_{bb}} \tag{6.4}$$

Using Maxwell's Reciprocal Theorem by replacing the displacement δ_{bn} by δ_{nb},

$$R_B = -\frac{\delta_{nb}}{\delta_{bb}} \tag{6.5}$$

The above equation forms the basis for determining the ordinates of influence lines since the displacements in both the numerator and denominator are due to the same unit vertical load acting at B. The value of δ_{nb} represents the displacement in the direction of R_B due to a unit load $P = 1$ travelling along the beam. The variation of δ_{nb} represents the elastic curve of the beam due to a unit load at B in the direction of R_B. Any of the methods of displacement computation may be used to find the shape of the elastic curve of the beam under a unit load at B. By dividing the ordinate of this curve by a constant factor $(-\delta_{bb})$ gives the ordinates of the graph representing R_B when a unit load $P = 1$ traverses the beam. This curve by definition describes the influence line of R_B.

The elastic curve δ_{nb} may be obtained using the conjugate beam method as shown in Fig. 6.3.

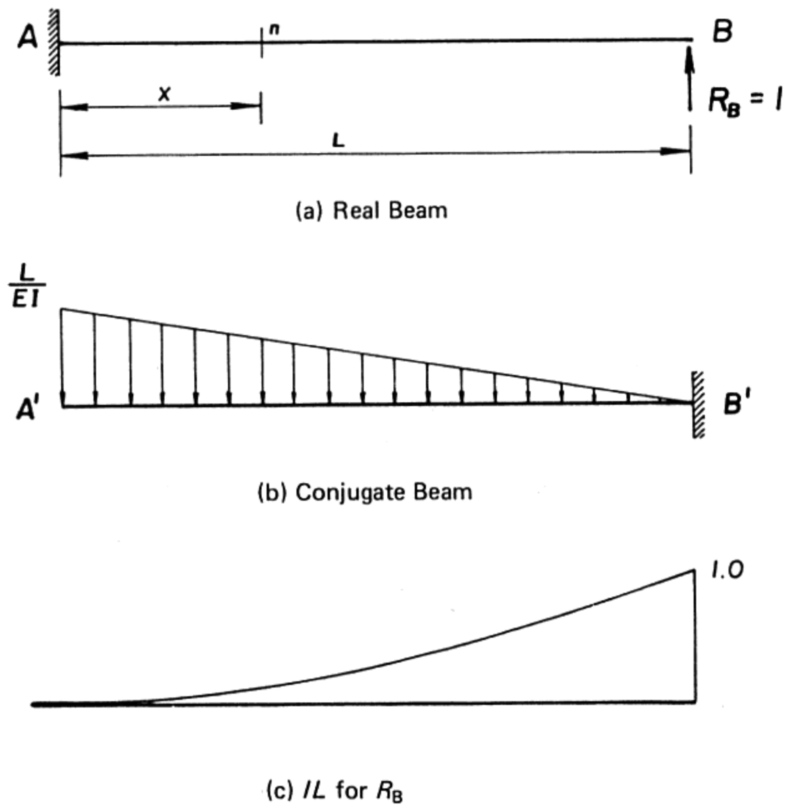


Figure 6.3

The displacement at any point n due to a unit load at B is

$$\delta_{nb} = -\frac{1}{EI}\left[\left(\frac{Lx}{2}\right)\left(\frac{2x}{3}\right) + \frac{(L-x)x}{2}\left(\frac{x}{3}\right)\right]$$

$$= -\frac{x^2}{6EI}(3L - x)$$

In a similar manner the displacement at B due to $R_B = 1$ is

$$\delta_{bb} = \frac{L^3}{3EI}$$

Hence, the equation of the influence line for the reaction at B becomes

$$R_B = -\frac{\delta_{nb}}{\delta_{bb}} = \frac{x^2(3L-x)}{2L^3}$$

The kinematic method may also be used to construct the influence line for internal stress resultants such as moments and shears. For the propped cantilever beam, the bending moment at midspan may be taken as the redundant reaction moment by introducing a hinge as shown in Fig. 6.4 if it is required to draw the influence line of the moment at midspan.

The redundant reaction moment is determined from compatibility conditions at the hinge such that the change of rotation of the two continuous sections to the right and to the left of the hinge must be zero. Thus

$$M_C\delta_{cc} + \delta_{nc} = 0$$

or

$$M_C = -\frac{\delta_{nc}}{\delta_{cc}} \qquad [6.6]$$

Therefore, the influence line for the bending moment at midspan will have the same shape as the deflection of the beam with a hinge at midspan. The conjugate beam method will be sufficient to determine the moment at C, which will be equal to δ_{cc} and the deflections at selected points along the span. For example, at point n $(0 < x < L/2)$

$$\delta_{nc} = \text{moment at } n \text{ of the conjugate beam}$$

$$= \frac{1}{EI}\left(\frac{2(L-x)x^2}{2L} + \frac{2x^3}{3L}\right)$$

$$= \frac{x^2}{EIL}\left(\frac{3L-x}{3}\right)$$

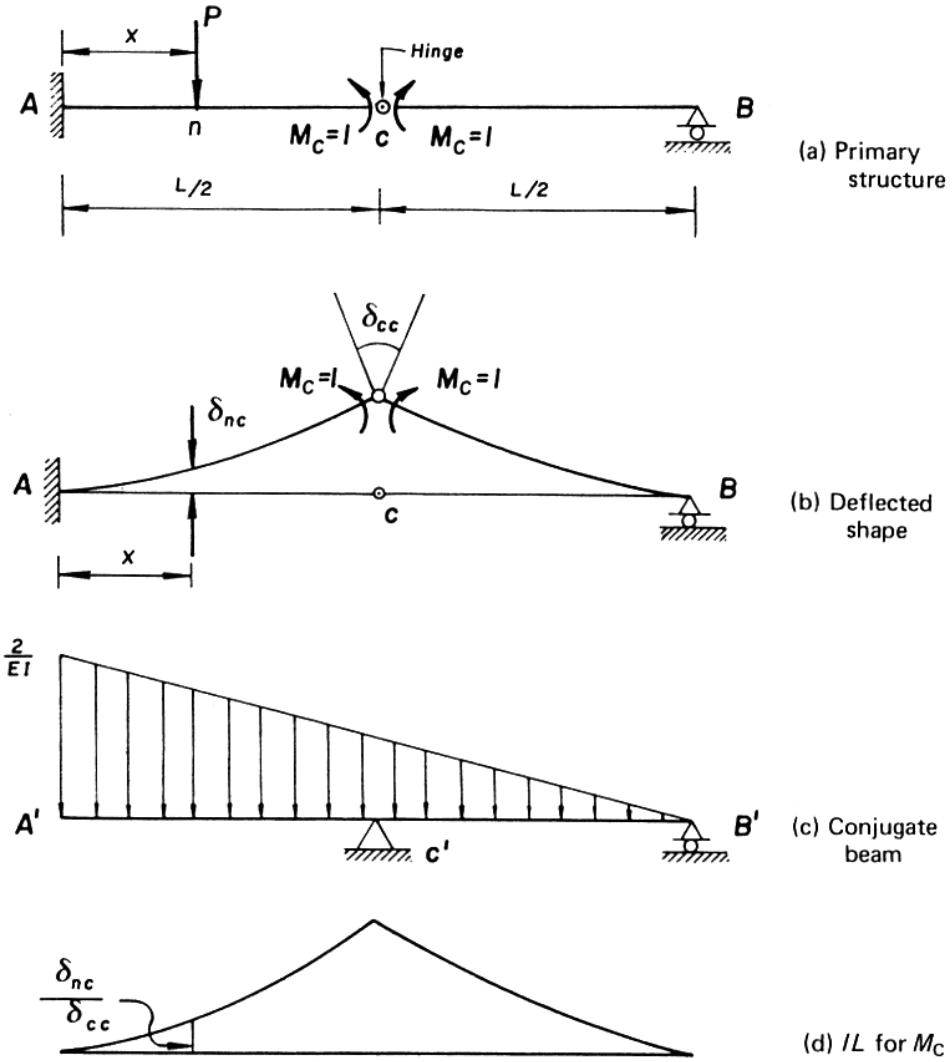


Figure 6.4

$$\delta_{cc} = \text{Reaction at C of the conjugate beam}$$

$$= \frac{1}{EI}\left[(2)(L)\left(\frac{1}{2}\right)\left(\frac{2}{3}L\right)\right] \Big/ \left(\frac{L}{2}\right)$$

$$= \frac{4L}{3EI}$$

The bending moment at point *n* is

$$M_C = \frac{\delta_{nc}}{\delta_{cc}}$$

$$= \frac{x^2(3L - x)}{4L^2} \text{ for } 0 < x < L/2$$

Similarly, when $x > L/2$

$$\delta_{nc} = \frac{x^2}{EIL}\left(\frac{3L - x}{3}\right) - \frac{4L}{3EI}(x - L/2)$$

The corresponding bending moment is

$$M_C = \frac{x^2(3L - x) - 4L^2x + 2L^3}{4L^2} \text{ for } L/2 < x < L$$

Notice that the above expressions are identical with those obtained by statical method.

When using the kinematic method to determine the influence line for the shear at the midspan of the propped cantilever beam, the beam may be made determinate by first cutting at C and inserting a two-bar linkage as shown in Fig. 6.5. This linkage system cannot carry shear force and distorts into a parallelogram shape thus establishing an equal rotation of the left and right tangents at C.

The displacement of the beam at point n $(0 < x < L/2)$ determined from the conjugate beam is

$$\delta_{nc} = -\frac{1}{EI}\left[(L - x)\left(\frac{x^2}{2}\right) + \left(\frac{x^2}{2}\right)\left(\frac{2x}{3}\right)\right]$$

$$= -\frac{x^2}{6EI}(3L - x) \quad \text{for } 0 < x < L/2$$

$$\delta_{cc} = M'_C = \frac{1}{EI}\left[(L)\left(\frac{L}{2}\right)\left(\frac{2L}{3}\right)\right] = \frac{L^2}{3EI}$$

The shear force when the load is at point n is

$$V_C = \frac{\delta_{nc}}{\delta_{cc}}$$

$$= -\frac{x^2(3L - x)}{2L^3}$$

Similarly, when $x > L/2$

$$\delta_{nc} = L^3/3EI - \frac{x^2}{6EI} - \frac{x^2}{6EI}(3L - x)$$

The corresponding shear force is

$$V_C = \frac{\delta_{nc}}{\delta_{cc}}$$

$$= 1 - \frac{x^2(3L - x)}{2L^3} \quad \text{for } L/2 < x < L$$

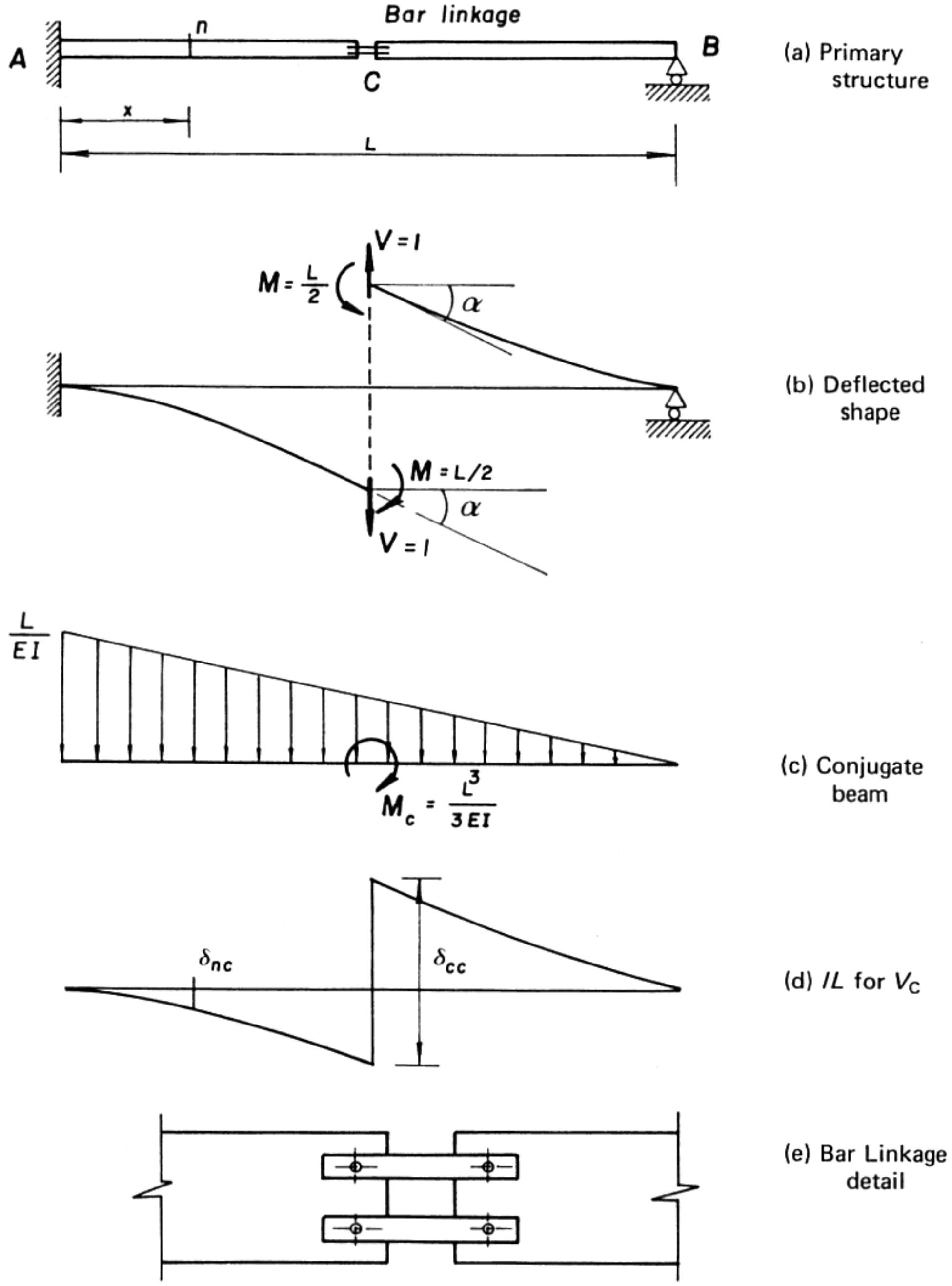


Figure 6.5

It may be concluded, therefore, that the kinematic method permits a simplified approach for the determination of the shape of the influence line for any action. This shape is identical to that of the elastic curve of the corresponding primary structure loaded by a unit force or moment at the point the redundant has been removed. This analogy, first recognised by Müller-Breslau, provides the most widely used and convenient method of computing the influence lines of indeterminate structures. The principle of Müller-Breslau may be stated as follows:

The ordinates of the influence lines of any function (such as reaction, moment, shear) of any structure is represented, to some scale, to those of the displacement curve which is obtained by removing the restraint corresponding to the relevant function imposing in its place a unit distortion such as rotation or a linear displacement.

6.3 INFLUENCE LINES FOR MULTIPLE REDUNDANT STRUCTURES

6.3.1 Influence Line for Bending Moment

Applying the Müller-Breslau principle to construct the influence line for moment at any point E between the support B and C of Fig. 6.6(a), a hinge is inserted at E so that the moment capacity of the beam is removed without impairing its shear capacity.

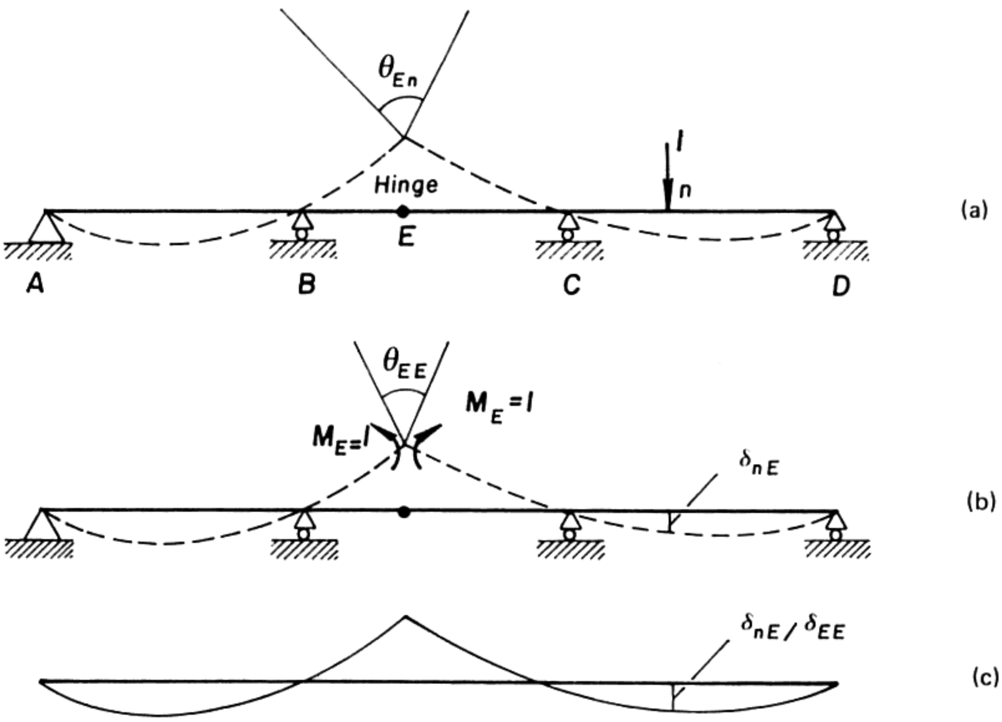


Figure 6.6

The beam is subjected to two force systems as shown in Fig. 6.6(b). A unit load is applied at n with the beam deflecting as shown in Fig. 6.6(a). The unit load is removed, and equal and opposite moments M are applied on either side of the hinge. The deflected shape of the beam is shown in Fig. 6.6(c). The shape of the deflected beam is to some scale the influence line of M_E. Thus

$$M_E = \frac{\theta_{En}}{\theta_{EE}} = \frac{\delta_{nE}}{\theta_{EE}} \qquad [6.7]$$

If $\theta_{EE} = 1$ radian, then

$$M_E = \delta_{nE} \tag{6.8}$$

Thus the influence line for M_E is obtained by dividing the ordinates of the deflected shape by θ_{EE} or by setting θ_{EE} equal to unity.

6.3.2 Influence Line for Shear Force

Let it be required to draw the influence line for shear at point E of the beam of Fig. 6.7(a). The beam is cut at E such that the shearing resistance is removed without impairing the flexural resistance. The beam is cut at E and a linkage system or slide device is inserted which cannot carry shear force and thus which permits a relative transverse deflection without introducing a change in slope of the left and right tangents at E.

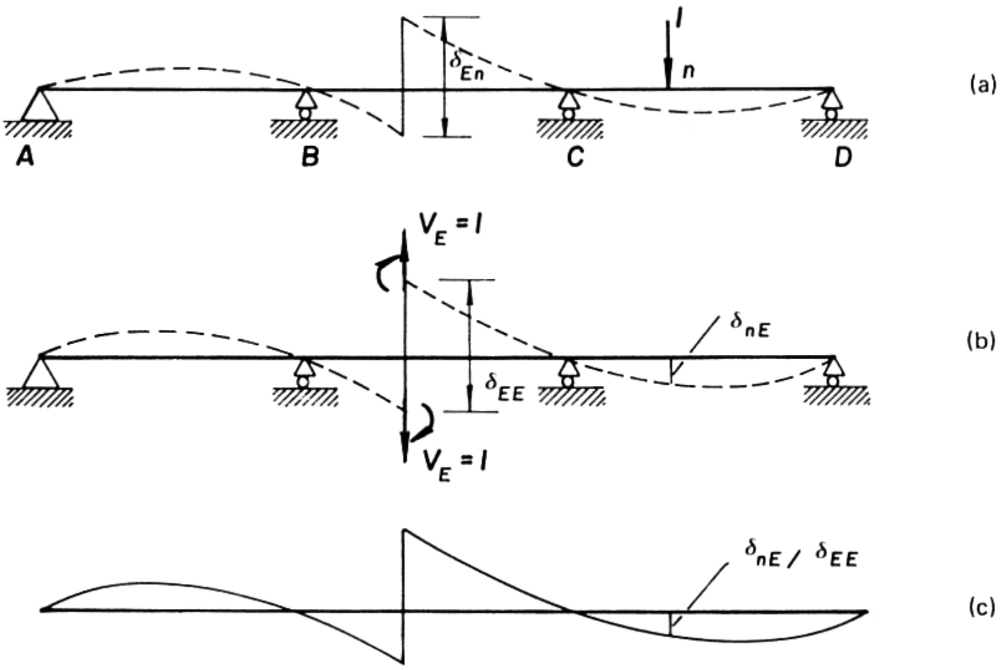


Figure 6.7

The beam is subjected to a unit load at n resulting with a vertical displacement δ_{En} at E. After removing the unit load, a pair of unit loads are applied as shown in Fig. 6.7(b). The shear force at E is given by

$$V_E = \frac{\delta_{En}}{\delta_{EE}} = \frac{\delta_{nE}}{\delta_{EE}} \tag{6.9}$$

If δ_{EE} = unity, then

$$V_E = \delta_{nE} \tag{6.10}$$

Therefore, the influence line for V_E is obtained by dividing the ordinates of the deflected shape by δ_{EE}.

EXAMPLE 6.1 *Compute ordinates of the influence line for the moment at B of the beam shown in Fig. 6.8. Use intervals of 2.5 m for span AB and 4.0 m for span BC. The moment of inertia is constant.*

Using the Müller-Breslau Principle, the capability of the beam to resist moment at the section for which the moment influence line is desired is removed by introducing a pin. Unit moments, say in kN m, are applied to the beam at each side of the pin at B. The modified beam, which will deflect as indicated by the dashed line, is shown in Fig. 6.8(b). According to the Müller-Breslau Principle, the various influence line ordinates are computed from the relation

$$M_B = \frac{\delta_{DB}}{\delta_{DD}}$$

Thus, the values of δ_{BB}, as well as the deflections at the necessary sections of the modified beam, may be evaluated using the conjugate beam method.

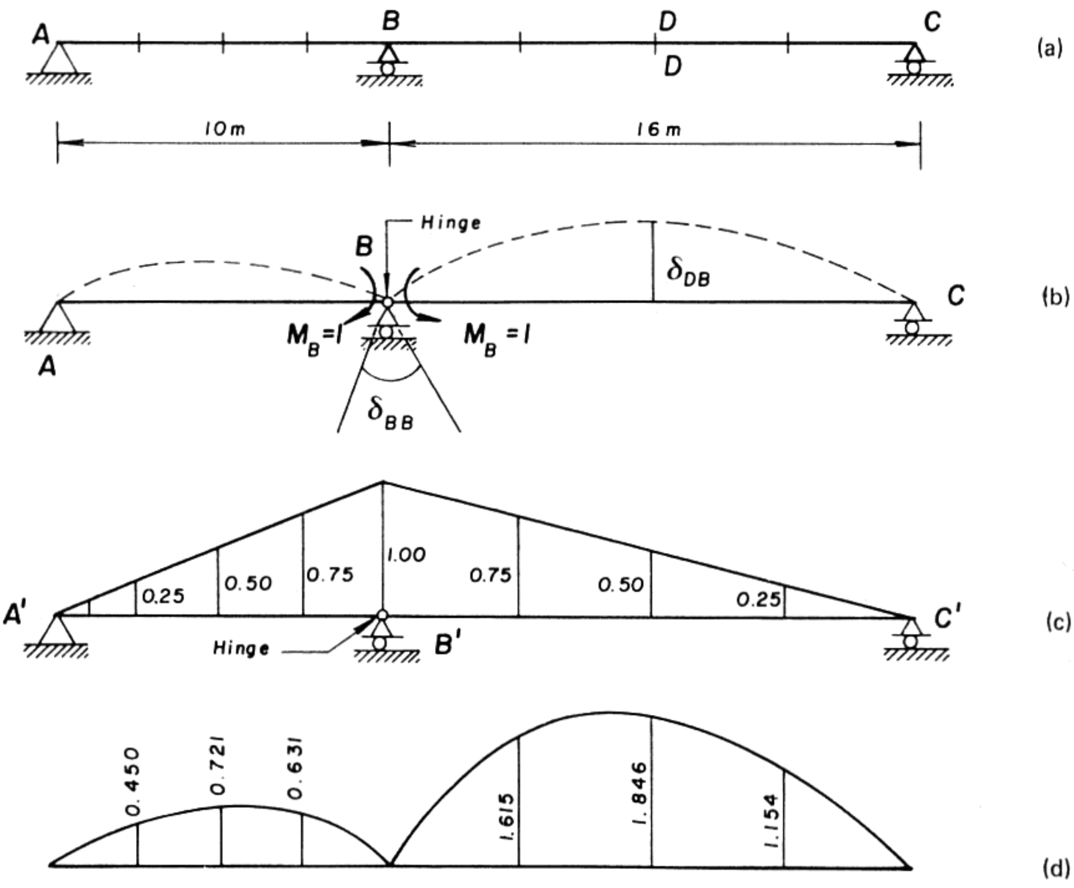


Figure 6.8

Determination of the reactions (*slopes*):

$$R_A = \frac{1}{3}\left(\frac{10 \times 1}{2}\right) = 1.667$$

$$R_B = \frac{2}{3}\left(\frac{10 \times 1}{2} + \frac{16 \times 1}{2}\right) = 8.667 = \delta_{BB}$$

$$R_C = \frac{1}{3}\left(\frac{16 \times 1}{2}\right) = 2.667$$

Determination of moments (*deflections*):

$$M_{2.5} = 1.667 \times 2.5 - \left(\frac{2.5 \times 0.25}{2}\right)\left(\frac{2.5}{3}\right) = 3.906$$

$$M_{5.0} = 1.667 \times 5.0 - \left(\frac{5.0 \times 0.50}{2}\right)\left(\frac{5.0}{3}\right) = 6.250$$

$$M_{7.5} = 1.667 \times 7.5 - \left(\frac{7.5 \times 0.75}{2}\right)\left(\frac{7.5}{3}\right) = 5.469$$

$$M_{10} = 0$$

$$M_{14} = 2.667 \times 12 - \left(\frac{12.0 \times 0.75}{2}\right)\left(\frac{12}{3}\right) = 14.00$$

$$M_{18} = 2.667 \times 8 - \left(\frac{8 \times 0.50}{2}\right)\left(\frac{8}{3}\right) = 16.00$$

$$M_{22} = 2.667 \times 4 - \left(\frac{4 \times 0.25}{2}\right)\left(\frac{4}{3}\right) = 10.00$$

The value of the influence line ordinate at each point is determined by dividing each moment by $\delta_{BB} = 8.667$. The resulting influence line is shown in Fig. 6.8(d).

The method of solution is first to assume that the fixed-end moment of 100 kN m exists at support B of the member BA with no other fixed-end moments being considered to exist.

6.3.3 Influence Lines by Moment Distribution

The Cross method of moment distribution may be used to obtain more easily influence lines for continuous beams and frames. The method is illustrated by the following example.

EXAMPLE 6.2 *Compute the ordinates of the influence line for the moment at B of Example 6.1 using the moment distribution method.*

The method of solution is first to assume that the fixed-end moment of 100 kN m exists at support B of the member BA with no other fixed-end moments being considered to exist. Using the moment distribution method, the moments are obtained from the balancing operation. Next, a fixed moment of 100 kN m is assumed at end B of member BC and again a similar operation is performed.

The moment distribution is shown in Table 6.2

Table 6.2

Joint	A	B		C
Member	AB	BA	BC	CB
K	$I/10$	$I/10$	$I/16$	$I/16$
DF	1.0	0.6154	0.3846	1.0
$M^F_{AB} = 100$	0	+100 −61.54	0 −38.46	0
	Σ	+38.46	−38.46	
$M^F_{BC} = 100$	0	0 −61.54	+100 −38.46	0
	Σ	−61.54	+61.54	

After finding the final moments due to the 100 kN m moment at every point where fixed-end moments can exist, the equation for M_{BA} in terms of the initial fixed-end moments is

$$M_{BA} = 0.3846M^F_{BA} + 0.6154MF^F_{BC}$$

The fixed-end moments for a 1.0 kN load placed successively at each of the points for which an influence line ordinate is desired, are computed below. The fixed-end moment for a propped cantilever loaded with one concentrated load P is

$$M^F_{BA} = \frac{Pab}{L^2}\left(a + \frac{b}{2}\right)$$

where a is measured from the pinned-end.

For span AB:

Table 6.3

$x = a$	$b = 10 - a$	$M^F_{AB} = ab(a + b/2)/L^2$
2.5	7.5	1.172
5.0	5.0	1.875
7.5	2.5	1.641

For span BC:

Table 6.4

$x = a$	$b = 16 - a$	$M^F_{BA} = ab(a + b/2)L^2$
4.0	12.0	1.875
8.0	8.0	3.000
12.0	4.0	2.625

Note: The ordinate x is measured with support C as the origin. The above values of the fixed moments are substituted in the equation above for M_{BA} and the influence line ordinates are computed in Table 6.5.

Table 6.5

Point, x(m)	M^F_{BA}	M^F_{BC}	$M_{AB} = 0.3846M^F_{BA} + 0.6154M^F_{BC}$
0	0		0
2.5	1.172		0.450
5.0	1.875		0.721
7.5	1.641		0.631
10.0		0	0
14.0		2.625	1.615
18.0		3.000	1.846
22.0		1.875	1.154
26.0			

The influence line for the moment at support B is shown in Fig. 6.8(d).

6.4 PROBLEMS

6.1 Draw the influence lines for the beam shown in Fig. P6.1 for the support reactions R_A and R_B.

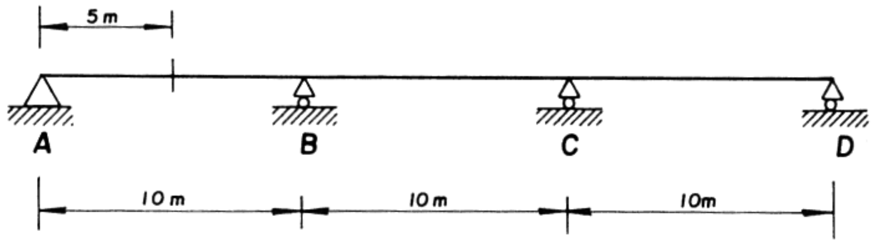


Figure P6.1

6.2 Draw the influence lines for the beam shown in Fig. P6.1 for the moments M_B and M_5.

6.3 Draw the influence lines for the beam shown in Fig. P6.1 for the shear S_5.

7. Introduction to Matrix Analysis

7.1 INTRODUCTION

After the introduction of high-speed computers, there has been a revolution in structural analysis, not only in the computational methods but also in the fundamental theorems. Since digital computers are ideally suitable for automatic computations of matrix algebra, it was found desirable to formulate the entire structural analysis in matrix notation. Matrix methods of structural analysis are based on the concept of replacing the actual structure by an equivalent analytical model consisting of discrete structural elements having known properties which can be expressed in matrix form. Matrices are useful in expressing structural theory and in producing an efficient means for carrying out numerical calculations.

Two methods have been formulated in matrix structural analysis: the flexibility and stiffness methods. It will not be possible in this textbook to develop the two matrix methods to sufficient depth. The methods are developed to the level of manual computation.

7.2 FORCE AND DISPLACEMENT MEASUREMENTS

It is evident that the overall description of the behaviour of a structure is accomplished through the dual consideration of force and displacement components at designated points. There are a number of ways of measuring a force applied to a structure or its displacement at designated points in a prescribed direction. Such points are commonly known as *node points*. The first step in the analysis of structures is to idealise the actual structure into a mathematical model which consists of distinct structural elements interconnected through node points. In this text the word *force* includes moment.

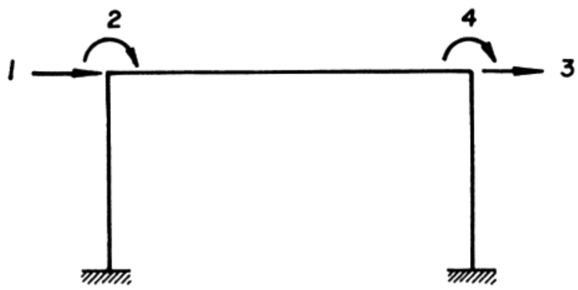


Figure 7.1

To designate the forces and displacements at the nodes of a given structure, a *coordinate system* is used to identify these measurements. For the frame shown in Fig. 7.1, for example, the system consists of four arbitrary coordinates which are identified by four numbered arrows shown at the specific nodes or joints. The forces are listed in column matrix $[P]$ and is referred to as a *force vector* and represents an ordered array of force measurements. For instance, the force vector for the frame of Fig. 7.1 is represented by

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \qquad [7.1]$$

Likewise, the coordinate displacement vector, having the same significance as in the force vector may be expressed as

$$[\Delta] = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \qquad [7.2]$$

In a similar manner, the forces and displacements at the nodes of a given element may be designated by listing in column matrices $[P]$ and $[\Delta]$, respectively. For the beam element of Fig. 7.2, for example, with direct forces at

Figure 7.2

node 1 and moments at node 2, the force vector is written as

$$[P] = \begin{bmatrix} P_{x1} \\ P_{y1} \\ M_{x2} \\ M_{y2} \end{bmatrix} \qquad [7.3]$$

and the displacement vector as

$$[\Delta] = \begin{bmatrix} u_1 \\ v_1 \\ \theta_{x2} \\ \theta_{y2} \end{bmatrix} \qquad [7.4]$$

A necessary step in the formation of the force and displacement vectors is the establishment of the node points and their location with respect to coordinate axes. At this stage it is necessary to define two sets of orthogonal coordinate systems. The first set is that of the structure, known as the *global axes*, and consists of a single coordinate system. The second set is that of the members or elements, known as the *local axes*, and consists of one coordinate system for each member. Since the members are in general differently oriented within a structure, these axes originating at member ends will usually be differently oriented from one element to the next. Global and local coordinates are illustrated in Fig. 7.3(a) for trusses and in Fig. 7.3(b) for frames.

When forces are applied to structures, displacements occur. Alternatively, when displacements are prescribed, node forces are necessary to produce them. The relationships that exist between applied forces and displacements play an important role in structural analysis. The force and displacement characteristics of a structure are usually described under definitions of *flexibility* and *stiffness* coefficients. The flexibility and stiffness coefficients depend on the force-displacement properties of the structure and the coordinate system used.

A simple illustration of such relationships is obtained by considering a linear elastic spring shown in Fig. 7.4. Single coordinate is indicated for the force and displacement measurements. The force P will stretch the spring thereby producing a displacement Δ at the end of the spring. The relationship between P and Δ can be expressed as

$$\Delta = fP \qquad [7.5]$$

In [7.5], f is the *flexibility coefficient* of the spring and is defined as the value of the displacement at node 1. In general, a flexibility coefficient is the value of the displacement at a point of the structure, in a given direction, due to a unit force applied at a second point in a second direction.

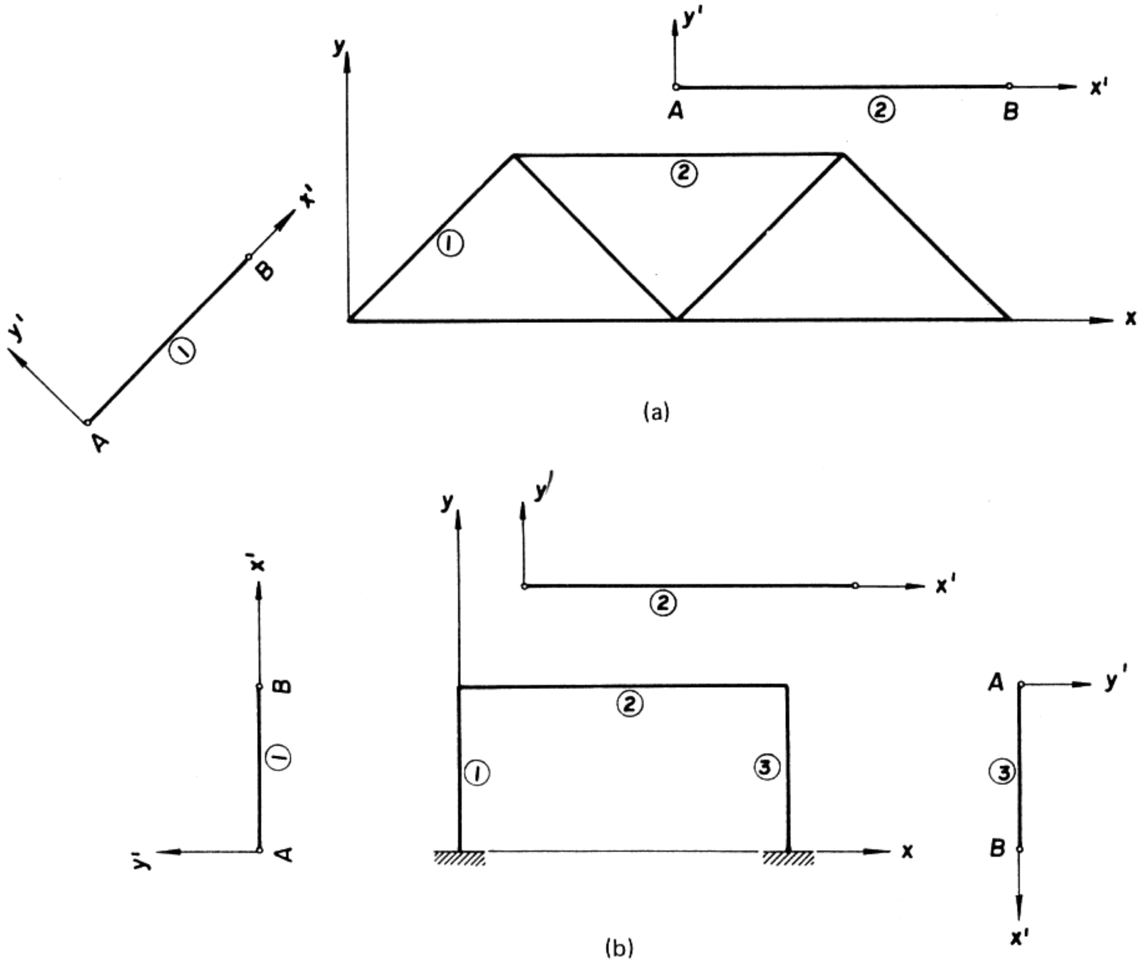


Figure 7.3

An alternative way is to establish a relationship between the force P and the displacement Δ for the spring of Fig. 7.4. The force P required to produce a displacement Δ units is determined from

$$P = k\Delta \tag{7.6}$$

In [7.6], k is the *stiffness coefficient* of the spring and is defined as the value of the force required at coordinate 1 to produce a unit displacement at 1. In general, a stiffness coefficient is the value of the force at a point of the structure, in a given direction, due to unit displacement applied at a second point in a second direction.

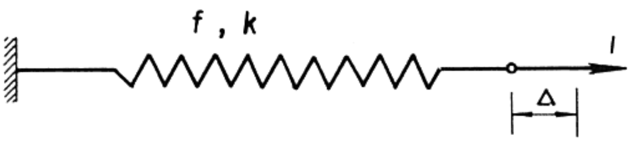


Figure 7.4

Comparison of [7.5] and [7.6] reveals that the flexibility and the stiffness of the spring are *inverse* to one another.

$$f = \frac{1}{k} = k^{-1}$$
$$k = \frac{1}{f} = f^{-1} \qquad [7.7]$$

Now consider a more general case consisting of an elastic structure, supported against rigid-body motion, and subjected to loads $P_1, P_2, \ldots, P_n$ acting at nodes $1, 2, \ldots, n$. The corresponding set of displacements is represented by Δ_1, $\Delta_2, \ldots, \Delta_n$. For linearly elastic systems, the principle of superposition is applicable. Therefore, the displacement Δ_i at node i is given by

$$\Delta_i = f_{i1}P_1 + f_{i2}P_2 + \ldots + f_{in}P_n \qquad [7.8]$$

or more generally,

$$\Delta_i = \sum_{j=1}^{j=n} f_{ij}P_j \qquad [7.9]$$

By definition, f_{ij} is the displacement produced at node i due to a unit load at node j ($P_j = 1$). The coefficients f_{ij}, which are the displacements due to unit loads, are known as flexibility coefficients.

In general, for n nodes, there will be n such displacements which may be written in a single matrix equation

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \ldots & f_{1n} \\ f_{21} & f_{22} & \ldots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1} & f_{n2} & \ldots & f_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} \qquad [7.10]$$

and which can be written in compact matrix form as

$$[\Delta] = [F]\,[P] \qquad [7.11]$$

where $[\Delta]$ is the column displacement matrix, $[F]$ is a square flexibility matrix and $[P]$ is a column load matrix (load vector). This equation is of the same type as [2.17].

Using matrix operation, one can solve the set of algebraic equations represented in [7.10] for forces in terms of displacements. In matrix notation

$$[P] = [F]^{-1}\,[\Delta] \qquad [7.12]$$

where $[F]^{-1}$ is the inverse of matrix $[F]$. It is noted that [7.12] has the same form as [7.6] since it expresses forces in terms of displacements. Consequently,

$$[F]^{-1} = [K] \qquad [7.13]$$

where $[K]$ is the stiffness matrix which is the inverse of the flexibility matrix. Thus

$$[P] = [K]\,[\Delta] \qquad [7.14]$$

The expanded form of [7.14] is

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} \qquad [7.15]$$

By definition, k_{ij} is the force required at node i to produce a unit displacement at node j only (zero displacements at all other nodes).

Flexibility coefficients for linear elastic behaviour have the property of reciprocity which may be expressed analytically as

$$f_{ij} = f_{ji} \qquad [7.16]$$

This equation defines symmetry of $[F]$. Since $[F]$ is symmetrical the inverse of a symmetric matrix will also become symmetrical. Therefore, [7.13] guarantees that the stiffness matrix $[K]$ will likewise be symmetrical. Consequently,

$$k_{ij} = k_{ji} \qquad [7.17]$$

To illustrate these matrices consider a simple cantilever beam of uniform cross section shown in Fig. 7.5(a). To determine the flexibility matrix, the influence coefficients must be determined by applying unit loads to the free end.

Due to axial load $N = 1$ (Fig. 7.5(b))

$$\delta_n = \frac{L}{EA}$$

$$\delta_{vn} = 0 \qquad [7.18]$$

$$\theta_n = 0$$

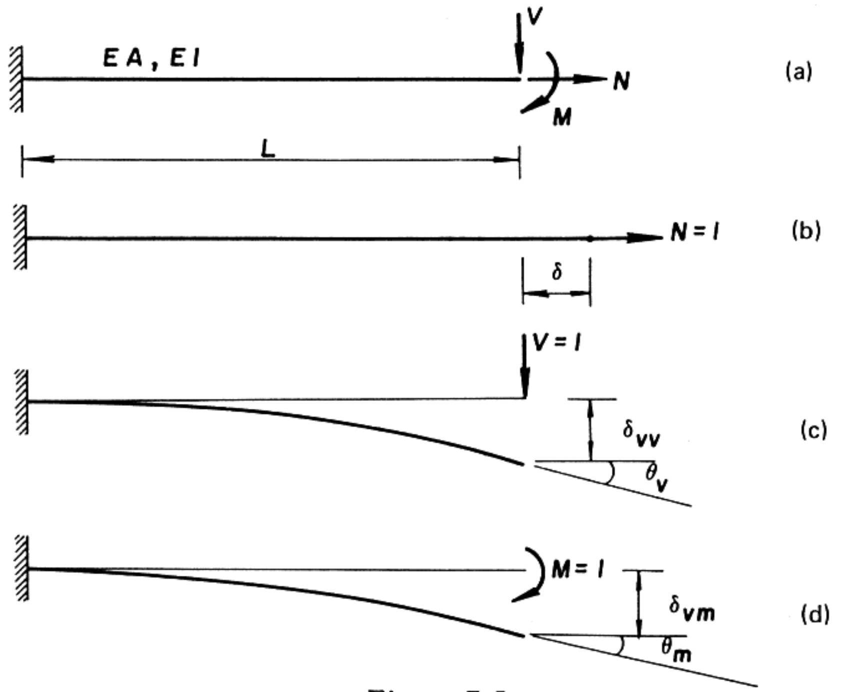


Figure 7.5

Due to vertical load $V = 1$ (Fig. 7.5(c))

$$\delta_n = 0$$

$$\delta_{vv} = \frac{L^3}{3EI} \qquad [7.19]$$

$$\theta_v = \frac{L^2}{2EI}$$

Due to moment $M = 1$ (Fig. 7.5(d))

$$\delta_n = 0$$

$$\delta_{vm} = \frac{L^2}{2EI} \qquad [7.20]$$

$$\theta_m = \frac{L}{EI}$$

The above results may be written in matrix form as

$$\begin{bmatrix} \delta_n \\ \delta_v \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{bmatrix} N \\ V \\ M \end{bmatrix} \qquad [7.21]$$

or, when written in compact matrix form

$$[\Delta] = [F]\,[P] \qquad [7.11]$$

In a similar manner the stiffness matrix may be determined by unit displacements as shown in Fig. 7.6.

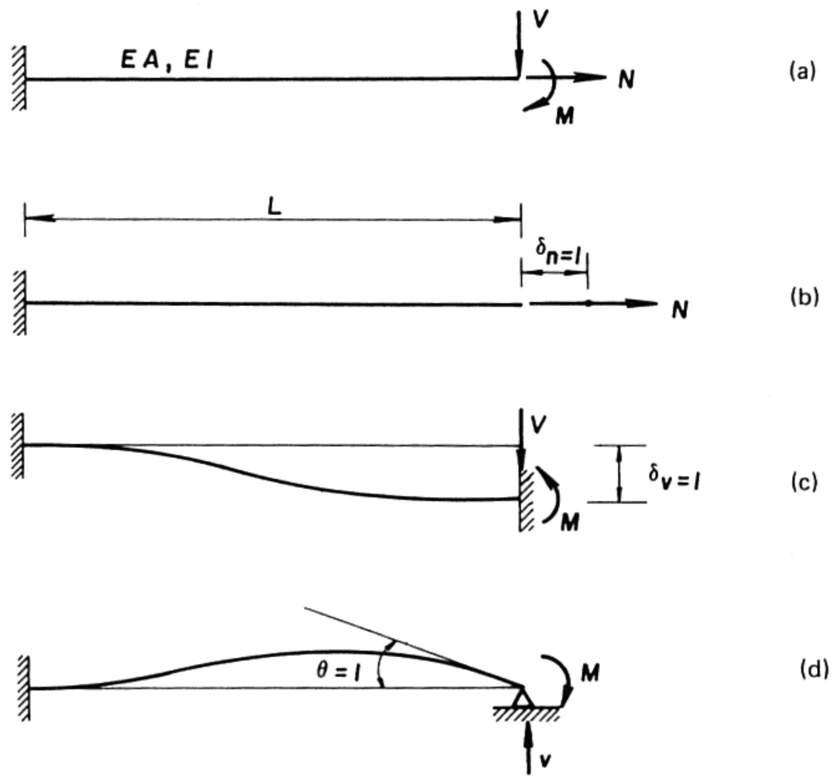


Figure 7.6

Due to unit axial displacement (Fig. 7.6(b))

$$N = \frac{EA}{L} \qquad [7.22]$$

Due to unit vertical displacement (Fig. 7.6(c))

$$V = \frac{12EI}{L^3} \qquad [7.23]$$

$$M = -\frac{6EI}{L^2}$$

Due to unit rotation (Fig. 7.6(d))

$$V = -\frac{6EI}{L^2}$$

$$M = \frac{4EI}{L} \qquad [7.24]$$

The above results may be written in matrix form as

$$\begin{bmatrix} N \\ \\ V \\ \\ M \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \delta_n \\ \\ \delta_v \\ \\ \theta \end{bmatrix} \qquad [7.25]$$

which may be written in compact matrix form as

$$[P] = [K]\,[\Delta] \qquad [7.26]$$

The results may be checked by matrix multiplication

$$[F]\,[K] = [K]\,[F] = [I] \qquad [7.27]$$

It is noted that both $[F]$ and $[K]$ are symmetric matrices which is the consequence of the reciprocal theorem.

7.3 THE FLEXIBILITY METHOD

The basic theory of the flexibility method is developed in this section, and the concepts are clarified by numerical examples. The development of the method rests on the basic principles of equilibrium of forces, compatibility and linear force-displacement relationships.

Consider a structure, which is idealised into a model consisting of distinct structural elements interconnected through node points, under the action of generalised external forces applied at the nodes $P_1, P_2, \ldots P_n$. These may be conveniently represented by a column matrix or force vector $[P]$

$$[P] = \{P_1, P_2 \ldots P_n\} \qquad [7.28]$$

Let it be assumed that the structure consists of m redundants which are forces to be determined, that is

$$[X] = \{X_1, X_2 \ldots X_m\} \qquad [7.29]$$

which are the redundant forces or reactions. If such redundants are removed, the structure becomes determinate and the internal forces are determined from conditions of equilibrium alone. In an indeterminate structure, the internal forces must also satisfy compatibility in addition to equilibrium. In dealing with an indeterminate structure with m redundants, the redundants are treated as additional loads on the statically determinate structure. It is assumed that the structure is composed of an assemblage of j simple elements. Internal forces exist in the structure at the node points. If the internal force members are represented by the vector $[S]$ where

$$[S] = \{S_1 \quad S_2 \quad \ldots \quad S_j\} \tag{7.30}$$

then, $[S]$ can be related to the applied loads $[P]$ and $[X]$ as

$$\begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_j \end{bmatrix} = [\mathrm{B}_0] \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} + [\mathrm{B}_1] \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X \end{bmatrix} \tag{7.31(a)}$$

which is written in compact notation as

$$[S] = [\mathrm{B}_0]\,[P] + [\mathrm{B}_1]\,[X] \tag{7.31(b)}$$

or using partitioned matrices

$$[S] = [\mathrm{B}_0 \mid \mathrm{B}_1] \begin{bmatrix} P \\ \hline X \end{bmatrix} \tag{7.31(c)}$$

where, in general, $[B_0]$ and $[B_1]$ are rectangular matrices whose elements are obtained from equilibrium conditions of the structure. For example, if P_i is taken as a unit load with all other loads including $[X]$ held at zero, the internal forces in the structure represent the coefficients corresponding to the ith column in the $[B_0]$ matrix. Likewise, the internal forces which result from a unit load X_j with all others held at zero represent the coefficients corresponding to the jth column of the $[B_1]$ matrix.

To formulate the compatibility condition, the principle of least work will be utilised which may be stated as: *The true values of the redundant forces are those which make the strain energy U of the strained structure a minimum.*

The strain energy is given as

$$U = \frac{1}{2}\,[S_1 \quad S_2 \quad \ldots \quad S_j] \begin{bmatrix} F_1 & & & & \\ & F_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & F_j \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_j \end{bmatrix} \tag{7.32}$$

which is written in compact matrix form as

$$U = \frac{1}{2}\,[S]^{\mathrm{T}}\,[F]\,\{S\} \tag{7.33}$$

In order to obtain the strain energy U in terms of the unknown $\{X\}$, substitute [7.31(b)] into [7.33]. In doing so, note that the transpose $\{S\}^T$ may be written from [7.31(b)] as

$$[S] = \begin{bmatrix} P \\ -- \\ X \end{bmatrix}^T [B_0 \mid B_1]^T$$

$$= [P \mid X] [B_0 \mid B_1]^T \qquad [7.34]$$

Substituting [7.31(b)] and [7.34] into [7.33]

$$U = \frac{1}{2} [P \mid X] [H] \begin{bmatrix} P \\ -- \\ X \end{bmatrix} \qquad [7.35]$$

where

$$[H] = [B_0 \mid B_1]^T [F] [B_0 \mid B_1] \qquad [7.36]$$

Since [P] and [X] are the applied and redundant forces, respectively, it is convenient to partition [H] to conform to the load vectors, thus,

$$U = \frac{1}{2} [P \mid X] \left[\begin{array}{c|c} H_{pp} & H_{px} \\ \hline H_{xp} & H_{xx} \end{array}\right] \begin{bmatrix} P \\ \cdots \\ X \end{bmatrix} \qquad [7.37]$$

After expanding [7.37]

$$U = \frac{1}{2} ([P] [H_{pp}] [P] + [P] [H_{px}] [X] + [X] [H_{xp}] [P] + [X] [H_{xx}] [X]) \qquad [7.38]$$

Utilising the theorem of least work and noting that the [H] matrix is symmetric gives

$$\frac{\partial U}{\partial X} = [H_{xp}] \; [P] + [H_{xx}] [X] = 0 \qquad [7.39]$$

from which the redundants are determined as

$$[X] = -[H_{xx}]^{-1} [H_{xp}] [P] \qquad [7.40]$$

Solving for [X] from [7.40] all internal forces can be determined from [7.31].

Summarising, the essential steps in applying the flexibility method to lead to the solution of structural problems may be stated as follows:

1. Idealise the structural problem to be analysed
2. Specify the redundant forces and identify the internal member forces

3. Find the $[B_0]$ matrix for unit values of external forces; only one external force must act at a time with all other forces held at zero
4. Find the $[B_1]$ matrix for unit values of redundant forces; only one redundant force must act at a time with all other forces held at zero
5. Find the flexibility matrix $[F]$ for all members following the sequential order of the member forces in $[B_0]$ and $[B_1]$
6. Formulate the $[H]$ matrix of [7.36]
7. Calculate the redundant forces $[X]$ from [7.40]
8. Calculate the internal forces $[S]$ from [7.31]

EXAMPLE 7.1 *Using the flexibility matrix method determine the bar forces in the truss with double diagonal system shown in Fig. 7.7(a). The area of all top chord is twice the area of all remaining members.*

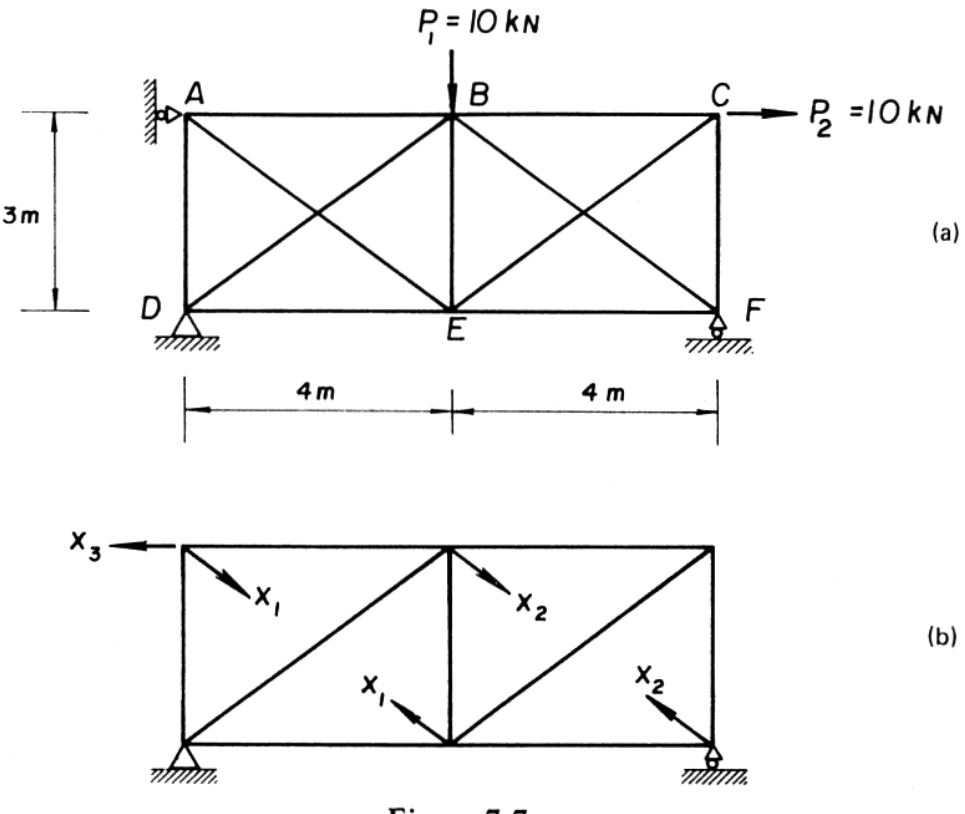


Figure 7.7

As can be easily seen, the truss is redundant to the second degree. For the selection of the redundant members several choices exist. Here members AE and BE and the reaction at A are taken as the redundants, then the truss is reduced to a determinate one as shown in Fig. 7.7(b).

To determine the $[B_0]$ matrix, P_1 and P_2 are set unit values one at a time with all other forces including the redundants held at zero, then calculate the internal forces in all members for each case. Thus,

$$
\begin{array}{cc}
 & \begin{array}{cc} P_1 = 1 & P_2 = 1 \end{array} \\
[B_0] = & \left[\begin{array}{cc}
0 & 0 \\
-0.667 & +0.500 \\
-0.500 & -0.375 \\
0 & 0 \\
+0.667 & 0.500 \\
0 & 0 \\
-0.833 & +0.625 \\
+0.833 & +0.625 \\
-0.500 & -0.375 \\
0 & 0 \\
0 & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\text{Member} \\ \text{AB} \\ \text{BC} \\ \text{CF} \\ \text{FE} \\ \text{ED} \\ \text{DA} \\ \text{DB} \\ \text{CE} \\ \text{BE} \\ \text{AE} \\ \text{BF}
\end{array}
$$

Similarly to determine the $[B_1]$ matrix the redundants X_1, X_2 and X_3 are set unit values one at a time with all other forces including the applied loads held at zero. The internal forces in each case are

$$
\begin{array}{cc}
 & \begin{array}{ccc} X_1 = 1 & X_2 = 1 & X_3 = 1 \end{array} \\
[B_1] = & \left[\begin{array}{ccc}
-0.80 & 0 & 1.0 \\
0 & -0.8 & +0.5 \\
0 & -0.6 & +0.375 \\
0 & -0.8 & 0 \\
-0.8 & 0 & -0.5 \\
-0.6 & 0 & 0 \\
1.0 & 0 & -0.625 \\
0 & 1.0 & -0.625 \\
-0.6 & -0.6 & 0.375 \\
1.0 & 0 & 0 \\
0 & 1.0 & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\text{Member} \\ \text{AB} \\ \text{BC} \\ \text{CF} \\ \text{FE} \\ \text{ED} \\ \text{DA} \\ \text{DB} \\ \text{CE} \\ \text{BE} \\ \text{AE} \\ \text{BF}
\end{array}
$$

The flexibility matrix for the members is

$$
[F] = \frac{1}{EA} \begin{bmatrix}
2 &&&&&&&&&& \\
& 2 &&&&&&&&& \\
&& 3 &&&&&&&& \\
&&& 4 &&&&&&& \\
&&&& 4 &&&&&& \\
&&&&& 3 &&&&& \\
&&&&&& 5 &&&& \\
&&&&&&& 5 &&& \\
&&&&&&&& 3 && \\
&&&&&&&&& 5 & \\
&&&&&&&&&& 5
\end{bmatrix}
$$

From [7.36]

$$[H] = [B_0 \mid B_1]^{\mathrm{T}} [F] [B_0 \mid B_1]$$

substituting and carrying out the matrix multiplications gives

$$[H] = \frac{1}{EA}\left[\begin{array}{cc|ccc} 11.111 & 1.792 & -5.400 & 7.031 & -3.125 \\ 1.792 & 6.25 & 2.200 & 3.675 & -5.250 \\ \hline -5.400 & 2.200 & 16.000 & 1.080 & -3.800 \\ 7.031 & 3.675 & 1.080 & 16.000 & -5.275 \\ -3.125 & -5.250 & -3.800 & -5.275 & 8.250 \end{array}\right]$$

$$= \left[\begin{array}{c|c} H_{pp} & H_{px} \\ \hline H_{xp} & H_{xx} \end{array}\right]$$

The redundants are determined using [7.40]

$$[X] = \begin{bmatrix} 16.000 & 1.080 & -3.800 \\ 1.080 & 16.000 & -5.275 \\ -3.800 & -5.275 & 8.250 \end{bmatrix}^{-1} \begin{bmatrix} 5.400 & 2.200 \\ 7.031 & 3.675 \\ -3.125 & -5.250 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Solving for the redundants,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4.617 \\ -3.743 \\ 9.883 \end{bmatrix}$$

The bar forces are determined using [7.31]

$$\begin{bmatrix} N_{\mathrm{AB}} \\ N_{\mathrm{BC}} \\ N_{\mathrm{CF}} \\ N_{\mathrm{FE}} \\ N_{\mathrm{ED}} \\ N_{\mathrm{DA}} \\ N_{\mathrm{DB}} \\ N_{\mathrm{CE}} \\ N_{\mathrm{BE}} \\ N_{\mathrm{AE}} \\ N_{\mathrm{BF}} \end{bmatrix} = \begin{bmatrix} 6.197 \\ 6.265 \\ -2.799 \\ 2.997 \\ 3.054 \\ -2.759 \\ -3.654 \\ 4.661 \\ -5.558 \\ 4.598 \\ 3.747 \end{bmatrix}$$

EXAMPLE 7.2 *Determine the shear force and bending moment values in the continuous beam of Fig. 7.8 using the flexibility matrix method.*

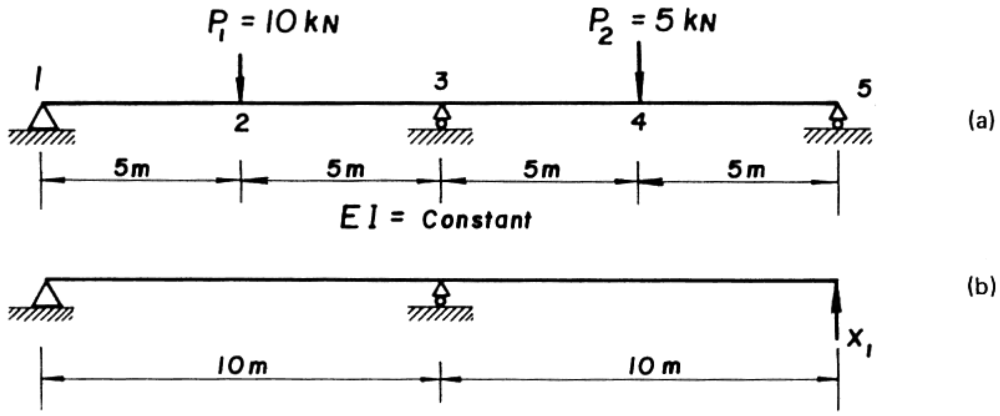


Figure 7.8

The beam is indeterminate to the first degree and the reaction at mode 5 is chosen as the redundant as shown in Fig. 7.8(b).

To determine the $[B_0]$ matrix, P_1 and P_2 are set unit values one at a time, thus

$$[B_0] = \begin{array}{cc} P_1 = 1 & P_2 = 1 \\ \end{array}$$

$$[B_0] = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0 \\ -0.5 & -0.5 \\ 2.5 & -2.5 \\ 0 & 1 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{array}$$

Similarly the $[B_1]$ matrix is determined setting $X_1 = 1$, thus

$$X_1 = 1$$

$$[B_1] = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \\ -1 \\ 10 \\ -1 \\ 5 \end{bmatrix} \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{array}$$

The member flexibility matrix from [7.20] neglecting the axial deformation is

$$[F] = \frac{0.833}{EI}\begin{bmatrix} \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & & & \\ & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & & \\ & & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & \\ & & & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} \end{bmatrix}$$

Using [7.36] to solve for the $[H]$ matrix

$$[H] = [B_0 \mid B_1]^{\mathrm{T}} [F] [B_0 \mid B_1]$$

$$[H] = \frac{0.833}{EI}\left[\begin{array}{cc:c} 25.0 & -37.5 & 75.0 \\ -37.5 & 150.0 & -325.0 \\ \hdashline 75.0 & -325.0 & 800.0 \end{array}\right]$$

$$= \left[\begin{array}{c:c} H_{pp} & H_{px} \\ \hdashline H_{xp} & H_{xx} \end{array}\right]$$

The redundant is determined using [7.40]. Hence,

$$X_1 = -\frac{1}{800} \quad [75 \quad -325] \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= 1.09375 \text{ kN}$$

The shear force and bending moment values at the indicated nodes are obtained using [7.31].

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{bmatrix} = \left[\begin{array}{cc:c} 0.5 & -0.5 & 1 \\ 0 & 0 & 0 \\ -0.5 & -0.5 & 1 \\ 2.5 & -2.5 & 5 \\ 0 & 1 & -1 \\ 0 & -5 & 10 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{array}\right] \left[\begin{array}{c} 10 \\ 5 \\ \hdashline 1.09375 \end{array}\right] = \begin{bmatrix} 3.594 \\ 0 \\ -6.406 \\ 17.969 \\ 3.906 \\ -14.063 \\ -1.094 \\ 5.469 \end{bmatrix}$$

7.4 THE STIFFNESS METHOD

As in the flexibility method, the stiffness method considers a structure as an assemblage of individual members. The connecting members are called *node points*. The fundamental difference in using the stiffness method is that the individual displacements of the nodes are taken as the unknowns. In the stiffness method the number of unknowns to be determined is the same as the degree of kinematic indeterminateness.

In this method, the first step is to derive the stiffness matrix for a component member by relating the member forces to the member deformations. In a similar manner the nodal forces must be related to the nodal displacements by the total stiffness matrix obtained from an assemblage of the stiffness matrices of the individual members. Finally, from equilibrium conditions the nodal forces obtained from the unknown nodal displacements must balance the externally applied nodal forces to find the total solution; that is, determining all unknown displacements, reactions and member forces. In developing the stiffness method, the same coordinate systems are used that were employed in the flexibility method.

Member Stiffnesses

The relationship between the forces acting at the nodes P_i and their corresponding nodal displacements forms the stiffness matrix approach. This relationship is given in matrix notation by [7.14] and in its generalised form by [7.15].

Consider a prismatic axial rod element *m* the ends of which are denoted as *i* and *j* as shown in Fig. 7.9.

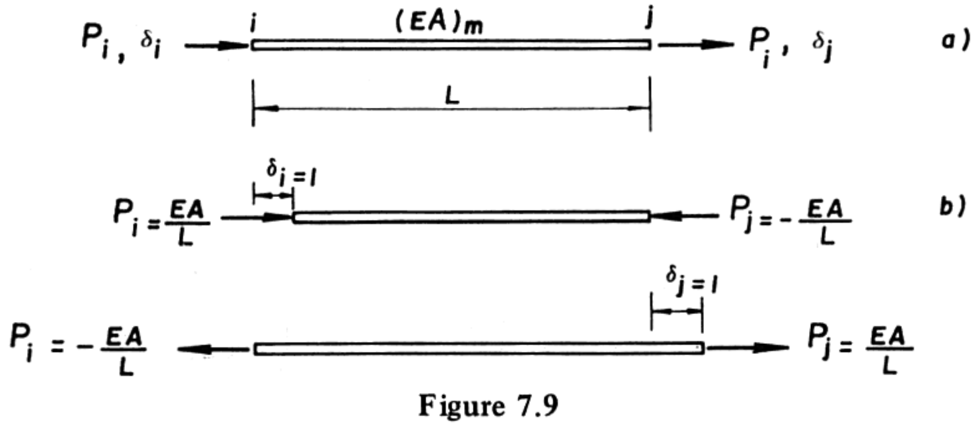


Figure 7.9

The relationship between the axial forces and the corresponding displacements of the rod is

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} \qquad [7.41]$$

The coefficients of the stiffness matrix are found by considering two distinct displacement states. The first state is to let the nodal coordinate displacement $\delta_i = 1$ as shown in Fig. 7.9(b) while holding the others at zero. Imposing equilibrium on the forces gives

$$P_i = -P_j = \left(\frac{EA}{L}\right)_m \tag{7.42}$$

The second state is similar, but distinct from the first. Following the same procedure as in the first state provides

$$P_j = -P_i = \left(\frac{EA}{L}\right)_m \tag{7.43}$$

Combining the results given by [7.42] and [7.43] into a single matrix equation yields the force–displacement relationship of an axial rod element

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \left(\frac{EA}{L}\right)_m \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} \tag{7.44}$$

Consider a prismatic beam element shown in Fig. 7.10. Using the same procedure used in obtaining [7.23] to [7.25] the force–displacement relationship for the given nodal coordinate system may be determined by assigning unit values to the displacements as shown in Fig. 7.10(b) and (c). The coefficients are shown for unit displacements at the end i; similar

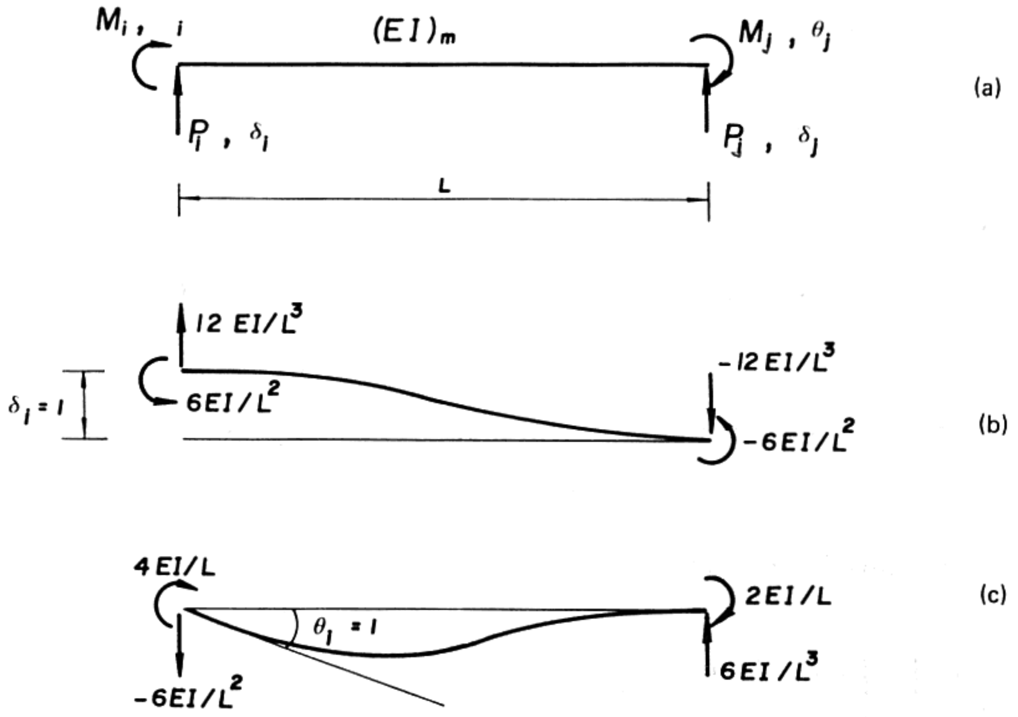


Figure 7.10

coefficients are obtained for unit displacements at end j. Thus

$$\begin{bmatrix} P_i \\ M_i \\ P_j \\ M_j \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_i \\ \theta_i \\ \delta_j \\ \theta_j \end{bmatrix} \qquad [7.45]$$

Transformation Matrices

If the properties of an element is known in terms of *local* axes, the transformations of these forces and displacements to the *global* coordinates is a necessary step in stiffness matrix formulation. Figure 7.11 shows member ij described by the two coordinate systems. The local coordinates are shown as x' and y' and global coordinates as x and y. In this text, the forces displacements and stiffness matrices with respect to local axes are identified by primes. The prime is omitted when written with respect to global axes.

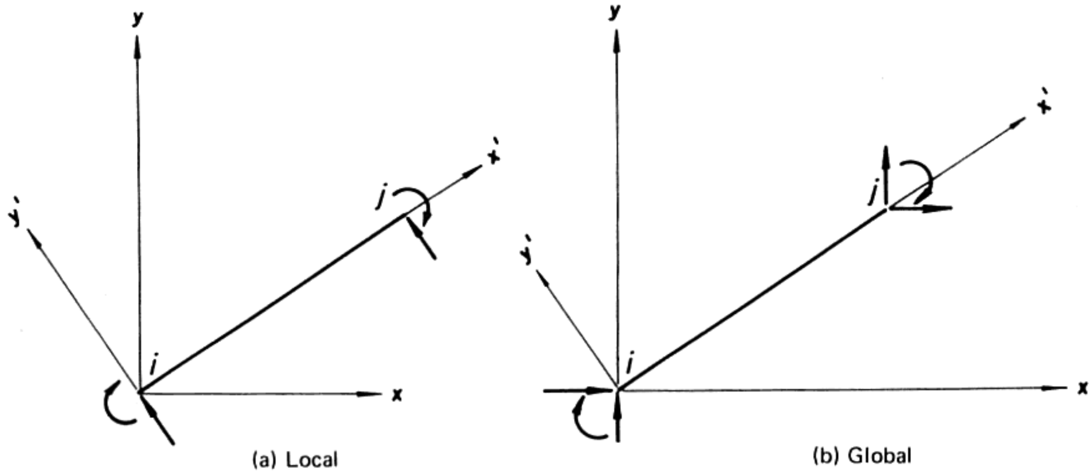


Figure 7.11

Referring to Fig. 7.11, the relationship between the quantities in the local and global axes for flexural members is established as

$$\begin{bmatrix} P'_{xi} \\ P'_{yi} \\ M'_i \\ \hline P'_{xj} \\ P'_{yj} \\ M'_j \end{bmatrix} = \left[\begin{array}{ccc|ccc} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} P_{xi} \\ P_{yi} \\ M_i \\ \hline P_{xj} \\ P_{yj} \\ M_j \end{bmatrix} \qquad [7.46]$$

and for axial members, after omitting M and θ, the relationship is

$$\begin{bmatrix} P'_{xi} \\ P'_{yi} \\ P'_{xj} \\ P'_{yi} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} P_{xi} \\ P_{yi} \\ P_{xj} \\ P_{yj} \end{bmatrix} \qquad [7.47]$$

Equations [7.46] and [7.47] may be written in compact matrix form

$$[P'] = [T]\,[P] \qquad [7.48]$$

where $[T]$ is the *rotational transformation matrix*, which is a function of the direction cosines between the two sets of axes, for the particular system shown. Solving [7.48] for $[P]$

$$\begin{aligned} [P] &= [T^{-1}]\,[P'] \\ &= [T]^{\mathrm{T}}\,[P'] \end{aligned} \qquad [7.49]$$

Such a matrix is called an *orthogonal matrix*, which may be defined as a square matrix having an inverse equal to its transpose.

If the displacements are denoted by $[\delta]$ then it follows that

$$[\delta'] = [T]\,[\delta] \qquad [7.50]$$

The transformation matrix $[T]$ may be applied to obtain the stiffness matrix in global coordinates. From the definition of stiffness, that is $[P] = [k]\,[\delta]$, it follows that

$$[P'] = [k']\,[\delta'] \qquad [7.51]$$

Substituting [7.48] and [7.50] into [7.51] and noting that $T^T = T^{-1}$ for orthogonal matrices

$$[T]\,[P] = [k']\,[T]\,[\delta]$$

or

$$\begin{aligned} [P] &= [T^{-1}]\,[k']\,[T]\,[\delta] \\ &= [T]^T[k']\,[T]\,[\delta] \\ &= [k]\,[\delta] \end{aligned} \qquad [7.52]$$

Hence, the transformed stiffness matrix is given by

$$[k] = [T]^T[k']\,[T] \qquad [7.53]$$

Using the relationship derived above, the stiffness matrix for axial members

(Fig. 7.11) in global coordinates will be

$$[k] = \left(\frac{EA}{L}\right)_m \left[\begin{array}{cc:cc} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ \hdashline -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{array}\right] \qquad [7.54]$$

where $\lambda = \cos\alpha$ and $\mu = \sin\alpha$. Similarly, for flexural members

$$[k] = \left(\frac{EI}{L^3}\right)_m \left[\begin{array}{ccc:ccc} 12\mu^2 & & & & & \\ -12\lambda\mu & 12\lambda^2 & & & \text{Symmetric} & \\ 6L\mu & -6L\lambda & 4L^2 & & & \\ \hdashline -12\mu^2 & 12\lambda\mu & -6L\mu & 12\mu^2 & & \\ 12\lambda\mu & -12\lambda^2 & 6L\lambda & -12\lambda\mu & 12\lambda^2 & \\ 6L\mu & -6L\lambda & 2L^2 & -6L\mu & 6L\lambda & 4L^2 \end{array}\right] \qquad [7.55]$$

Assembly of Element Matrices

It is important to form the total assemblage nodal stiffness matrix of a structure from the stiffness matrices of the separate structural elements. This involves only simple additions when all element stiffness matrices have been expressed in the same global coordinate system.

Consider the axial member system shown in Fig. 7.12 with a total of three possible joint displacements one for each node. The members have individual stiffness constants $(EA/L)_1$ and $(EA/L)_2$ as shown in the figure.

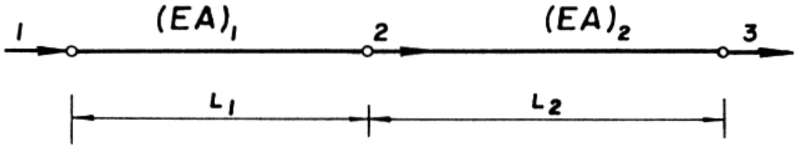


Figure 7.12

The order of the stiffness matrix for the assemblage will be 3 x 3. The individual member stiffness matrices are:

$$[k_1] = \left(\frac{EA}{L}\right)_1 \begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$[k_2] = \left(\frac{EA}{L}\right)_2 \begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \end{array} \qquad [7.56]$$

The assembled stiffness matrix for the complete system can be formed by superposition of the individual element stiffnesses contributing to each nodal point. Thus, the assembled stiffness matrix for the system shown in Fig. 7.12 becomes

$$[K] = \begin{array}{c} \left[\begin{array}{c:c:c} \left(\frac{EA}{L}\right)_1 & -\left(\frac{EA}{L}\right)_1 & 0 \\ \hdashline -\left(\frac{EA}{L}\right)_1 & \left(\frac{EA}{L}\right)_1 + \left(\frac{EA}{L}\right)_2 & -\left(\frac{EA}{L}\right)_2 \\ \hdashline 0 & -\left(\frac{EA}{L}\right)_1 & \left(\frac{EA}{L}\right)_2 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad [7.57]$$

In a similar manner, it may be concluded that the order of the global stiffness matrix of a system is equal to the total number of degrees of freedom of the system. The order of the matrix may be expressed as the sum of the unknown displacements f and the prescribed (support) displacements s. After reordering the rows and columns to separate the elements corresponding to the supports from the remainder, the rearranged stiffness matrix may be written as

$$\begin{bmatrix} P_f \\ \hdashline P_s \end{bmatrix} = \left[\begin{array}{c:c} K_{ff} & K_{fs} \\ \hdashline K_{sf} & K_{ss} \end{array}\right] \begin{bmatrix} \Delta_f \\ \hdashline \Delta_s \end{bmatrix} \qquad [7.58]$$

Method of Solution

Expanding [7.58] and noting that the support displacements, $\{\Delta_s\} = 0$

$$[P_f] = [K_{ff}]\,[\Delta_f] \qquad [7.59(a)]$$

$$[P_s] = [K_{sf}]\,[\Delta_f] \qquad [7.59(b)]$$

The vectors of all unknown nodal displacements (at unsupported nodes) are obtained from [7.59(a)]

$$[\Delta_f] = [K_{ff}]^{-1}\,[P_f] \qquad [7.60]$$

When $[\Delta_f]$ has been found from [7.60], the support reactions by substitution of the results in [7.59(b)] will be

$$[P_s] = [K_{sf}]\,[K_{ff}]^{-1}\,[P_f] \qquad [7.61]$$

The internal force in any element m may be obtained by substituting the calculated degrees of freedom for that element, designated by $[\Delta_m]$, into the element stiffness matrix $[k_m]$. Thus, the joint force component acting on that element becomes

$$[P_m] = [k_m]\,[\Delta_m] \qquad [7.62]$$

For the case of an axial member shown in Fig. 7.11 the member force, denoted by P_m, is found to be

$$P_m = P_{xj} \cos \alpha + P_{yj} \sin \alpha \qquad [7.63]$$

but

$$\begin{aligned} P_{xj} &= \frac{EA}{L}((\delta_{xj} - \delta_{xi})\cos^2\alpha + (\delta_{yj} - \delta_{yi})\cos\alpha\sin\alpha) \\ P_{yj} &= \frac{EA}{L}((\delta_{xj} - \delta_{xi})\cos\alpha\sin\alpha + (\delta_{yj} - \delta_{yi})\sin^2\alpha) \end{aligned} \qquad [7.64]$$

Rearranging and writing in matrix form, [7.64] may be written as

$$P_{\mathrm{m}} = \left(\frac{EA}{L}\right)_{\mathrm{m}} [\cos\alpha \quad \sin\alpha] \begin{bmatrix} \delta_{xj} - \delta_{xi} \\ \delta_{yj} - \delta_{yi} \end{bmatrix} \qquad [7.65]$$

Similarly, for the case of a beam element [7.45] and the true nodal displacements are used and the internal forces for the beam element taken to be the shear and bending moments are

$$\begin{bmatrix} V_i \\ M_i \end{bmatrix} = \left(\frac{EI}{L^3}\right)_m \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \end{bmatrix} \begin{bmatrix} \delta_{yi} \\ \theta_i \\ \delta_{yi} \\ \theta_j \end{bmatrix} \qquad [7.66]$$

When these are external loads acting between the joints of a beam element the concept of equivalent loads must be adopted. The member action is then computed by adding the effects of the member end deformation to the fixed-end actions produced by the loads. In a similar manner, the support reactions are computed by adding the fixed-end effects of the loads. Thus [7.14] may be written as

$$[P] = [K]\,[\Delta] - [P^F] \qquad [7.67]$$

where $[P^F]$ is the load vector of the fixed-end actions.

EXAMPLE 7.3 *Determine the bar forces, using the stiffness matrix method, of the truss shown in Fig. 7.13. EA = constant.*

Member data for the truss

Member	Member ends		Member properties		*Direction cosines*	
	i	*j*	*A*	*L*	*cos α*	*sin α*
1	A	B	*A*	*L*	0	1
2	A	C	*A*	1.155*L*	–0.5	–0.866
3	A	D	*A*	1.4146*L*	0.707	–0.707

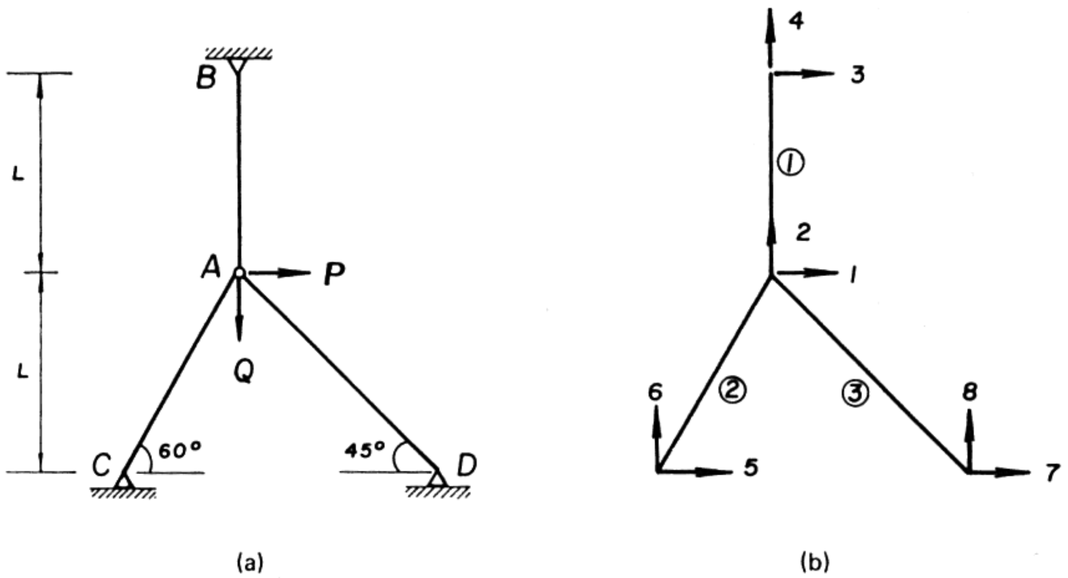


Figure 7.13

The member stiffnesses oriented in global coordinate system are obtained from [7.54].

Member AB

$$[k_1] = \frac{EA}{L}\begin{array}{c}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}$$

Member AC

$$[k_2] = \frac{EA}{1.155L}\begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix}\begin{array}{c}1\\2\\5\\6\end{array}$$

$$k_3 = \frac{EA}{1.414L}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}\begin{array}{c}1\\2\\7\\8\end{array}$$

After assemblage of the element stiffness matrices, the global stiffness